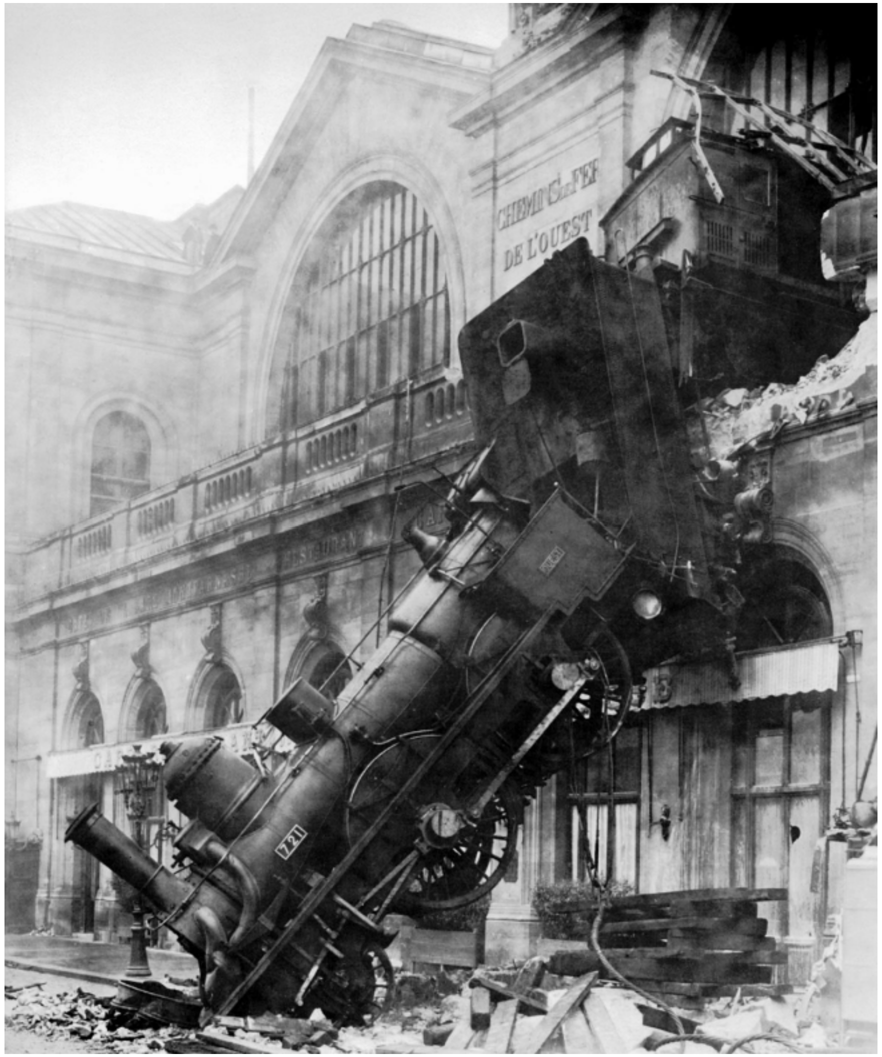
Reliability and risk analysis

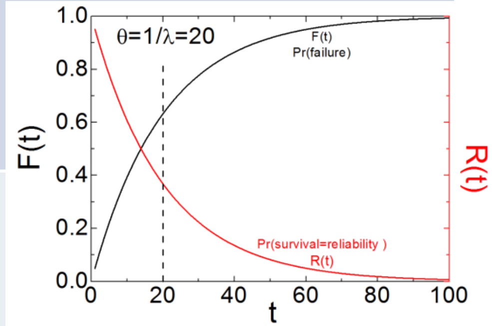
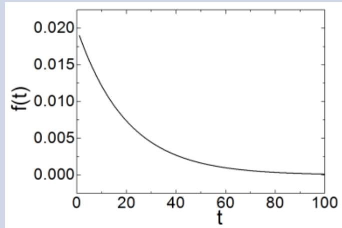
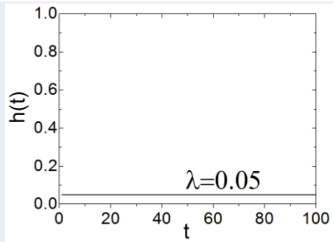
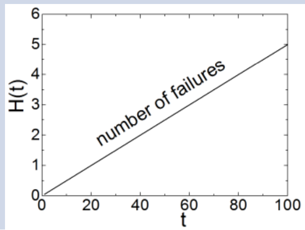


Luis Carlos Pardo . C2.4

Summary

0.- Remeber
1.- Bayes theorem
2.- Maximum likelihood method
3.- Estimation of reliability parameters from tests
4.- Confidence limits of parameters
5.- Accelerated life testing
6.- Determination of distribution models
7.- Empirical determination of survivor function
8.- Reliability growth
9.- Strength-stress models

Summary of important quantities

	Description	example	graph
cdf $R_T(t)$	Reliability function or Survival function	$R_T(t) = e^{-\lambda t}$	
cdf $F_T(t)$	Failure probability Lifetime distribution function	$F_T(t) = 1 - e^{-\lambda t}$	
pdf $f_T(t)$	Probability to fail Between t and t+dt	$f_T(t) = \lambda e^{-\lambda t}$	
Pdf $\lambda(t)$ $h(t)$	Failure rate or Hazard function	$h(t) = \lambda$	
cdf $H(t)$	Number of failures	$H(t) = \lambda \cdot t$	

Summary of important relations

$$R_T(t) = 1 - F_T(t)$$

$$f_T(t) = \frac{dF_T(t)}{dt}$$

$$F_T(t) = \int_0^t f_T(t)\,dt$$

$$f_T(t)$$

failure probability

$$h(t) = \frac{f_T(t)}{R(t)}$$

failure ratio

$$H(t)$$

$$h(t) = \frac{dH(t)}{dt}$$

$$H(t) = \int_0^t h(t)\,dt$$

$$h(t)$$

Summary of important relations

$$F_T(t) = 1 - e^{-\lambda t}$$

Failure probability

$$f_T(t) = \frac{dF_T(t)}{dt}$$

$$F_T(t) = \int_0^t f_T(t)\,dt$$

$$f_T(t)$$

Failure probability at t

$$h(t) = \frac{f_T(t)}{R(t)}$$

Number of failures

$$H(t) = \lambda \cdot t = \#\ failures$$

$$h(t) = \frac{dH(t)}{dt}$$

$$H(t) = \int_0^t h(t)\,dt$$

failure ratio

$$h(t) = \lambda = \frac{failures}{time}$$

Summary

1.- Bayes theorem
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And finally: Bayes theorem

From the “multiplication” of probability:

$$prob(A,B) = prob(A) \cdot prob(B|A)$$

$$prob(A,B) = prob(B) \cdot prob(A|B)$$

dividing the two eqations we get:

Bayes theorem

$$prob(A|B) = \frac{prob(B|A)\,prob(A)}{prob(B)}$$

That help us to reverse probabilities...

Bayes theorem

$$prob(H \mid D) = \frac{prob(D \mid H)\, prob(H)}{prob(D)}$$

Posterior prob(H|D):
What you want to know is the probability that your hypothesis is true given the data

Likelihood (or L) prob(D|H):
What you know is yout hypothesis. And therefore you can calculate the probability that your data “gathers” around your hypothesis

Prior prob(H):
You might want to include any prioir information about your hypothesis

Evidence (E) prob(D):
Is “simply” a normalization factor... we will come back when doing model selection

Bayes theorem

Likelihood
Probability that our data describes the hypothesis

Prior
Our forehand knowledge

$$P(H\,|\,D) = \frac{P(D\,|\,H)\cdot P(H)}{P(D)}$$

T. Bayes.

Posterior
Probability that the hypothesis is true given the experimental data

Evidence
Normalization factor

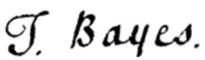
In our case is very simple...

Bayes theorem

Likelihood
Probability that our data describes the hypothesis

Maximum ignorance Prior

$$P(H \mid D) \propto \frac{P(D \mid H) \cdot P(H)}{P(D)}$$



We only care about proportionality

Posterior
Probability that the hypothesis is true given the experimental data

Thus, assuming a maximum ignorance prior:

$$prob(H \mid D) \propto prob(D \mid H) = L$$

But what is exactly H, in practice?

It is a function and the parameters inside it

$$H_i(a,b) = a + bx_i$$

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<u>Parameter estimation</u>

We would like to estimate the best estimate of a quantity, given some data.
Let's define the posterior as:

$$prob\left(X \mid \{data\}\right) = P$$

The best estimate of X is given by a maximum in its probability and thus

$$\left.\frac{dP}{dX}\right|_{X_0} = 0$$

We want now to expand P as a Taylor series, since P varies too fast we take the logarithm of P

$$L = \ln(P) = \ln\left(prob\left(X \mid \{data\}\right)\right)$$

The Taylor expansion of L is:

$$L = L(X_0) + \frac{1}{2}\left.\frac{d^2L}{dX^2}\right|_{X_0}\left(X - X_0\right)^2 + \ldots$$

Taking the first term we get for P:

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[\frac{1}{2}\left.\frac{d^2L}{dX^2}\right|_{X_0}\left(X - X_0\right)^2\right]$$

Let's compare with a gaussian

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[\frac{\left(X - X_0\right)^2}{2\sigma^2}\right]$$

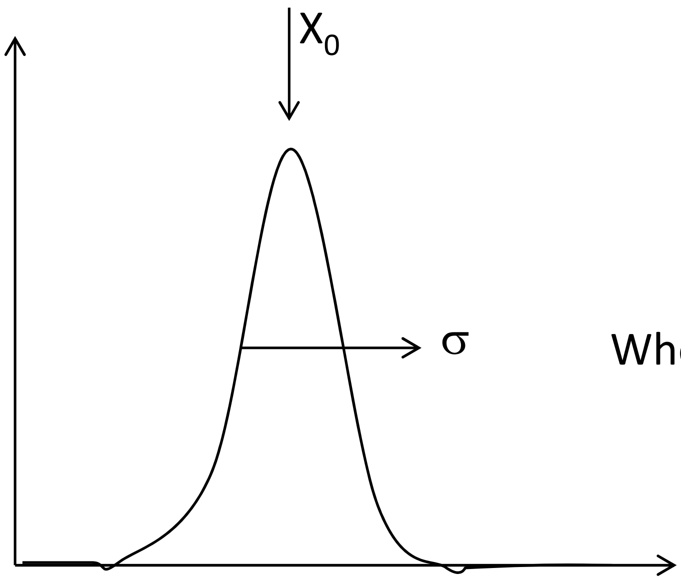
Comparing

Taking the first term we get for P:

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[\frac{1}{2}\frac{d^2L}{dX^2}\bigg|_{X_0}\left(X - X_0\right)^2\right]$$

Let's compare with a gaussian

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[-\frac{\left(X - X_0\right)^2}{2\sigma^2}\right]$$



$$\sigma = \left(-\frac{d^2L}{dX^2}\bigg|_{X_0}\right)^{-1/2}$$

Where $\frac{d^2L}{dX^2}\Big|_{X_0}$ is always negative, so: no problem ;-)

$$X = X_0 \pm \sigma$$

Value plus error comes from a Taylor series

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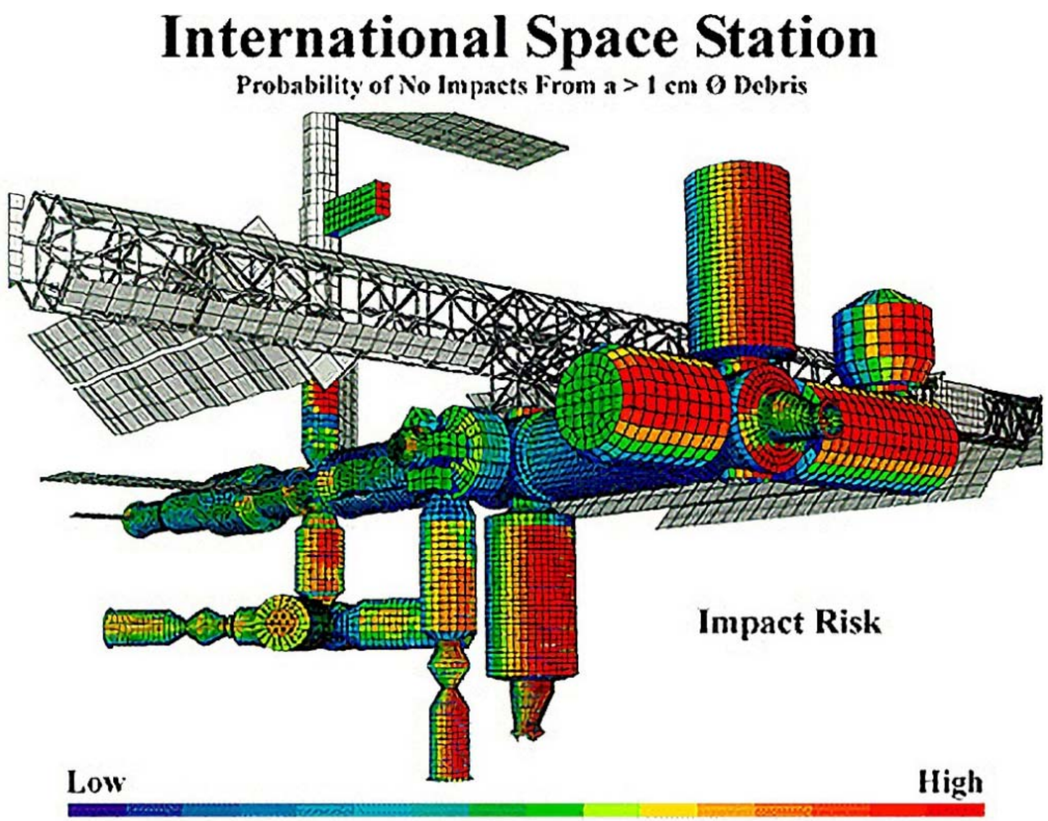
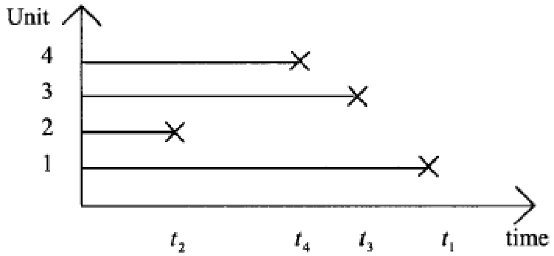
Let's perform tests to quantify risk analysis...
before sending the ISS to the space!!!!



Types of tests:

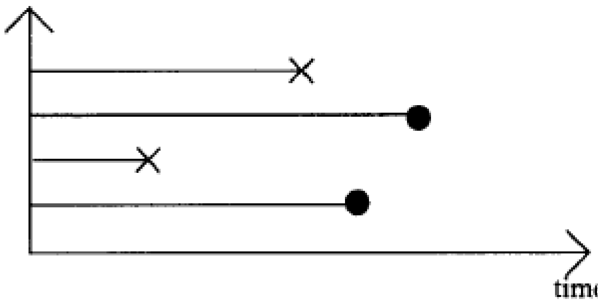
Let's assume that we perform a test on n=4 components

Complete data sets

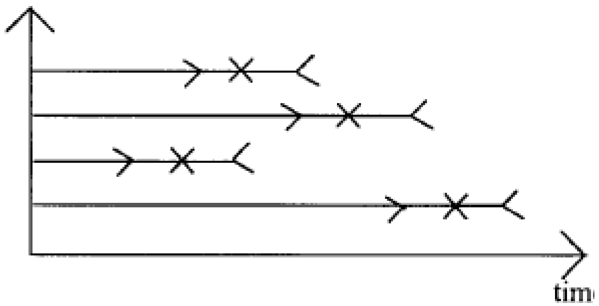


Censored data sets

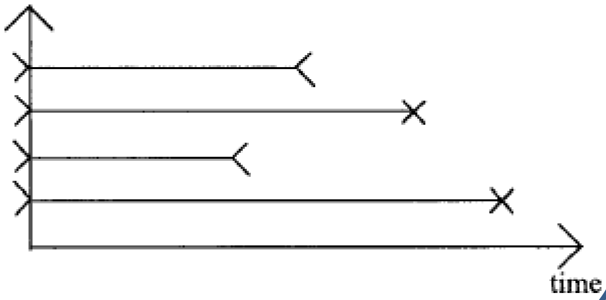
Right censored



Interval censored



Leftcensored

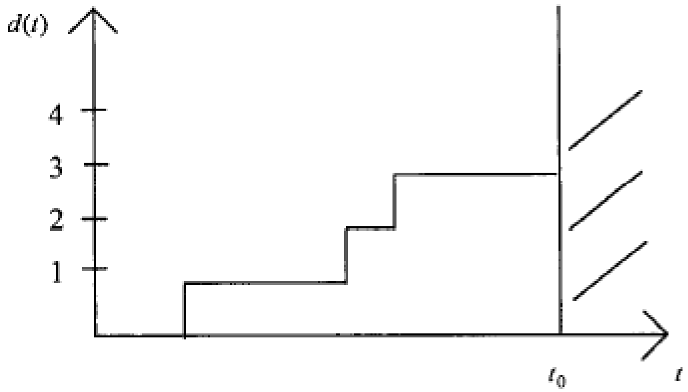


Types of tests:

Test plans

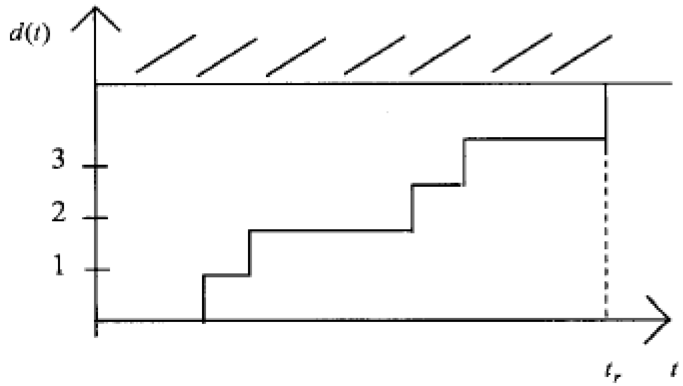
Type I

Testing up to a given time t0

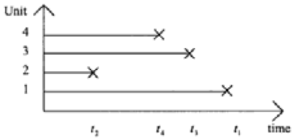


Type II

Testing up to a given r-th failure



They can also be with (R) or witgout (W) replacement



Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***non-censored*** data

We calculate the **likelihood** that

$$L = \prod_{i=1}^{n} \lambda e^{-\lambda t_i} = \lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda t_2} \cdot \lambda e^{-\lambda t_3} \ldots \lambda e^{-\lambda t_n} = \lambda^n e^{-\lambda T} \leftarrow \boxed{T = \sum_{i=1}^{n} t_i}$$

we take the **logarithm**

$$\ln L = \ln\left(\lambda^n e^{-\lambda T}\right) = n \ln \lambda - \lambda T$$

we make the **derivative**

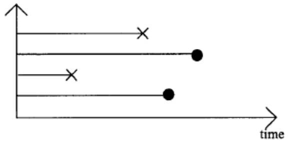
$$\frac{\partial \ln L}{\partial \lambda} = \ln\left(\lambda^n e^{-\lambda T}\right) = \frac{\partial\left(n \ln \lambda - \lambda T\right)}{\partial \lambda} = \frac{n}{\lambda} - T$$

we **set it to zero** (find the maximum)

$$\frac{n}{\hat{\lambda}} - T = 0$$

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{n}{T}$$



Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type I (until t0)

We calculate the **likelihood** that

$$L = \prod_{i=1}^{r} \underbrace{f(t_i \mid \lambda)}_{\text{failure}} \prod_{i=r+1}^{n} \underbrace{R(t_0 \mid \lambda)}_{\text{time right-censored}} = \lambda^r e^{-\lambda \sum_{i=1}^{r} t_i} e^{-\lambda(n-r)t_0} = \lambda^r e^{-\lambda T}$$

$$T = \sum_{i=1}^{n} t_i + (n-r)t_0$$

we take the **logarithm**

$$\ln L = \ln\left(\lambda^r e^{-\lambda T}\right) = r \ln \lambda - \lambda T$$

we make the **derivative**

$$\frac{\partial \ln L}{\partial \lambda} = \ln\left(\lambda^r e^{-\lambda T}\right) = \frac{\partial\left(r \ln \lambda - \lambda T\right)}{\partial \lambda} = \frac{r}{\lambda} - T$$

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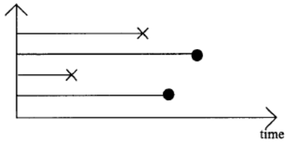


Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type II (until r units fail)

We calculate the **likelihood** that

$$L = \underbrace{\prod_{i=1}^{r} f\left(t_i \mid \lambda\right)}_{\text{failure}} \underbrace{\prod_{i=r+1}^{n} R\left(t_i \mid \lambda\right)}_{\text{right-censored}} = \lambda^r e^{-\lambda \sum_{i=1}^{r} t_i} e^{-\lambda \sum_{i=r+1}^{n} t_i} = \lambda^r e^{-\lambda T}$$

$$T = \sum_{i=1}^{n} t_i + (n-r) t_r$$

we take the **logarithm**

$$\ln L = \ln\left(\lambda^r e^{-\lambda T}\right) = r \ln \lambda - \lambda T$$

we make the **derivative**

$$\frac{\partial \ln L}{\partial \lambda} = \ln\left(\lambda^r e^{-\lambda T}\right) = \frac{\partial\left(r \ln \lambda - \lambda T\right)}{\partial \lambda} = \frac{r}{\lambda} - T$$

we **set it to zero** (find the maximum)

$$\frac{r}{\hat{\lambda}} - T = 0$$

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{r}{T}$$

...BUT YOU DO NOT WANT TO STOP PRODUCTION until r components fail

Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type I (until t0)

WITH REPLACEMENT

n=1

Let's do it for a single component **n=1** that is replaced each time it fails

$$L = \lambda e^{-\lambda t_1} \lambda e^{-\lambda(t_2 - t_1)} \lambda e^{-\lambda(t_3 - t_2)} \ldots \lambda e^{-\lambda(t_0 - t_r)} = \lambda^r e^{-\lambda t_0}$$

n>1

... for **n** components

$$L = \lambda^r e^{-\lambda n t_0} = \lambda^r e^{-\lambda T} \leftarrow \boxed{T = n t_0}$$

doing the same calculations as before...

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{r}{T}$$

...BUT YOU DO NOT WANT TO STOP PRODUCTION until r components fail

Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type II (until r-th failure)

WITH REPLACEMENT

n=1

Let's do it for a single component **n=1** that is replaced each time it fails

$$L = \lambda e^{-\lambda t_1} \lambda e^{-\lambda(t_2 - t_1)} \lambda e^{-\lambda(t_3 - t_2)} \ldots \lambda e^{-\lambda(t_0 - t_r)} = \lambda^r e^{-\lambda t_0}$$

n>1

... for **n** components

$$L = \lambda^r e^{-\lambda n t_0} = \lambda^r e^{-\lambda T} \quad \leftarrow \boxed{T = n t_r}$$

doing the same calculations as before...

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{r}{T}$$

summarizing: We want to keep the notation $\hat{\lambda} = \frac{r}{T}$

so, we change the meaning of T...

not censored, no replacement $T = \sum_{i=1}^{n} t_i$

	Censored Type I (unit t0)	Censored Type II (unit r-th failure)
Without replacement	$T = \sum_{i=1}^{n} t_i + (n-r) t_0$	$T = \sum_{i=1}^{n} t_i + (n-r) t_r$
With replacement	$T = n t_0$	$T = n t_r$

Summary

1.- Bayes theorem

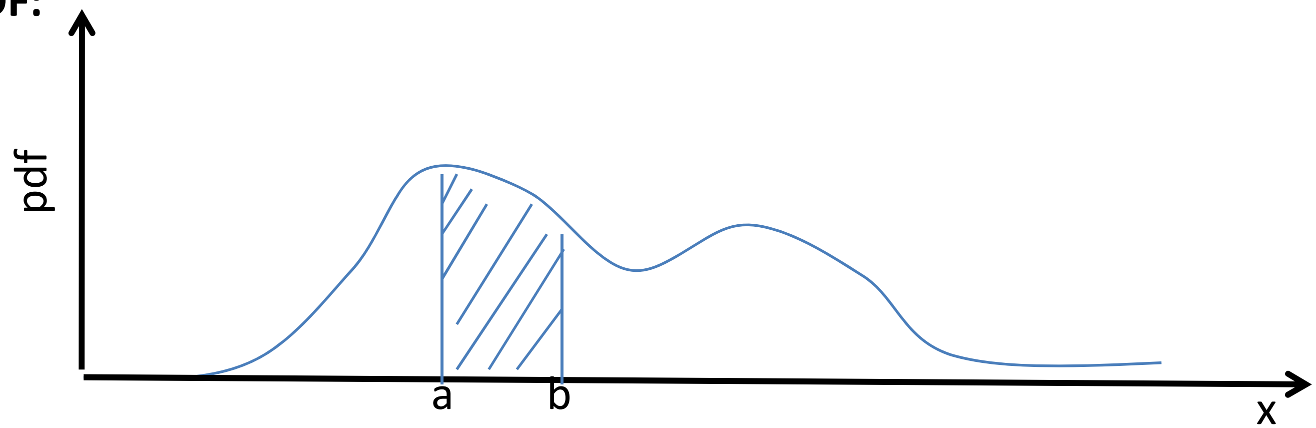
2.- Maximum likelihood method

3.- Estimation of reliability parameters from tests

4.- Confidence limits of parameters:
normal distribution

Confidence intervals: the general case

Let's consider a PDF:



Relation with probabilities

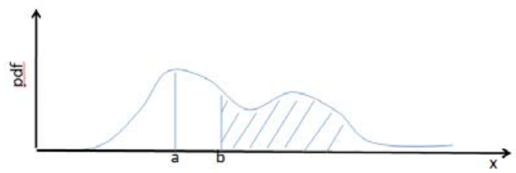
$$p(a < x < b) = \int_a^b f(x)dx$$

Therefore

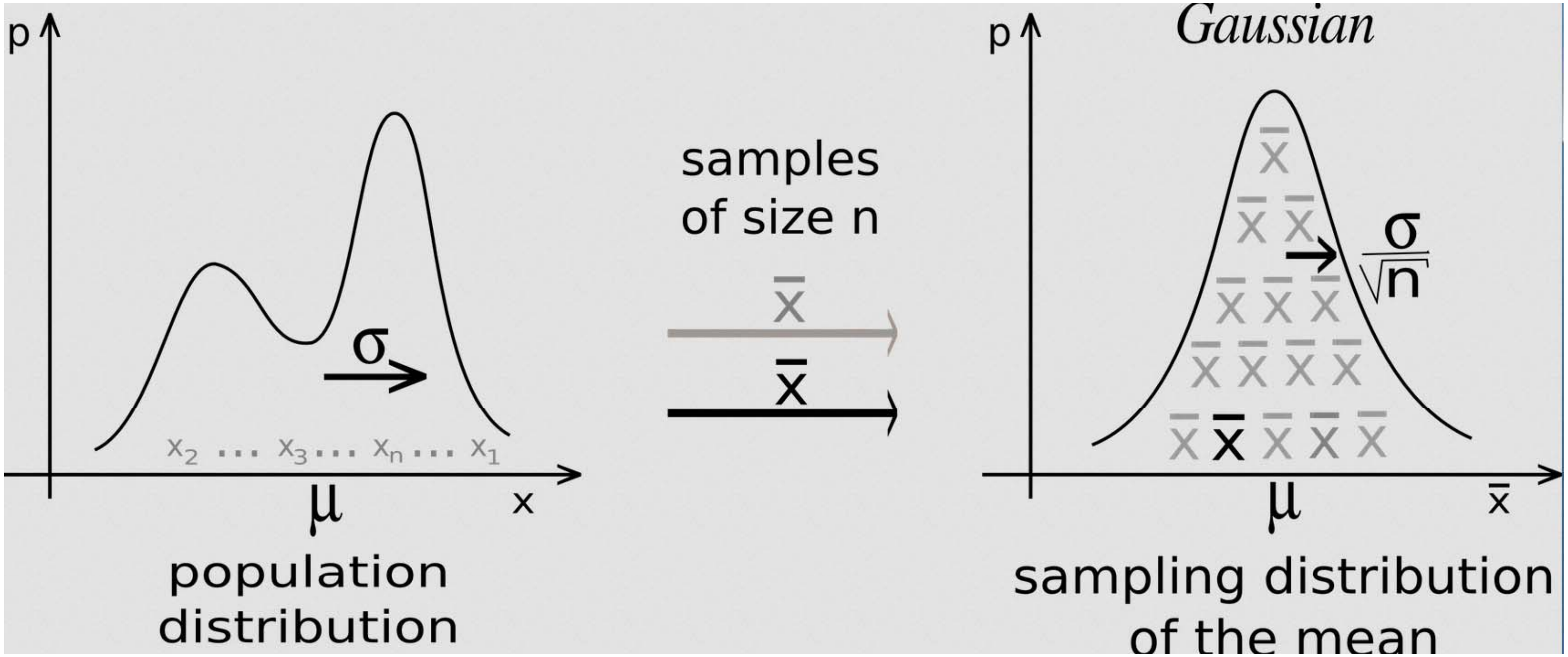
$$\int_{-\infty}^{\infty} f(x) = 1$$

P-value for b (one side)

$$p(x > b) = \int_b^{\infty} f(x)dx$$



Central limit theorem



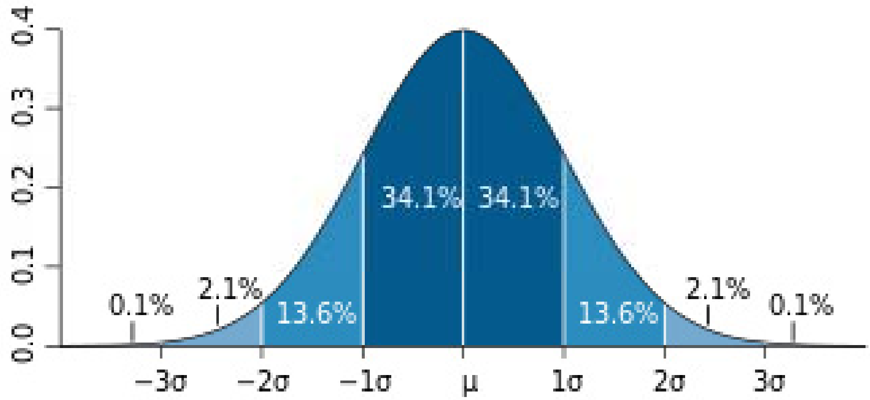
The distribution of the mean tends to be gaussian when increasing n

no matters the pdf that originated the mean!!!!

This is the good and the bad thing from statistics is based on the normal PDF!!!!

Standard Normal distribution

Standard Normal distribution (μ=0, σ=1)



$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

Therefore, defining a "z-value" for the whole population as

$$z = \frac{x-\mu}{\sigma_x}$$

Therefore, defining a "z-value" for n units

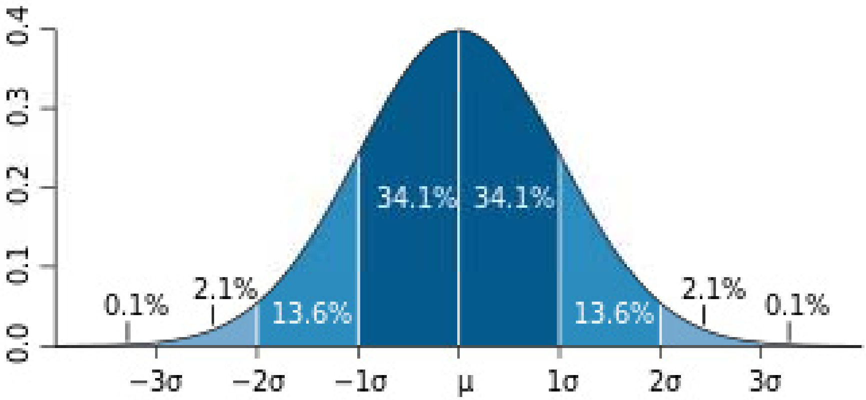
$$z = \frac{\bar{X}-\mu}{\sigma_x / \sqrt{n}}$$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2\right) \Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)$$

Z follows a **standard** normal distribution function...

Confidence intervals: the normal case

Normal distribution



$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(-\frac{(x-\mu)^2}{2\sigma_x^2} \right)$$

$$f(x) \equiv N(\mu, \sigma)$$

Relation with probabilities

$$p(-\sigma_x < x < \sigma_x) = \int_{-\sigma}^{\sigma} f(x)dx = 68.2\%$$

P-value for normal distribution for a value a: is related with the variance

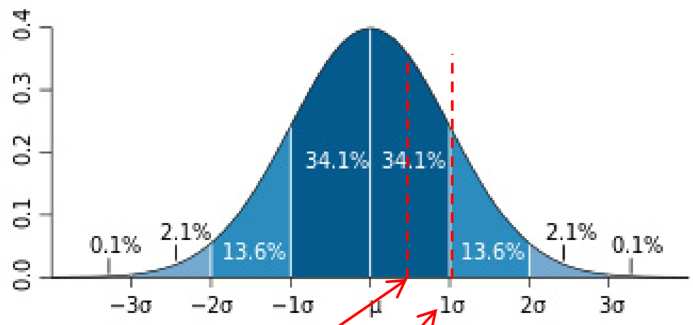
$$p(-\sigma_x < x < \sigma_x) = \int_{-\sigma}^{\sigma} f(x)dx = 68.2\%$$

So we have a relationship between confidence intervals and probability!

There is a 68,2% of chances that your real value is inside your “error “

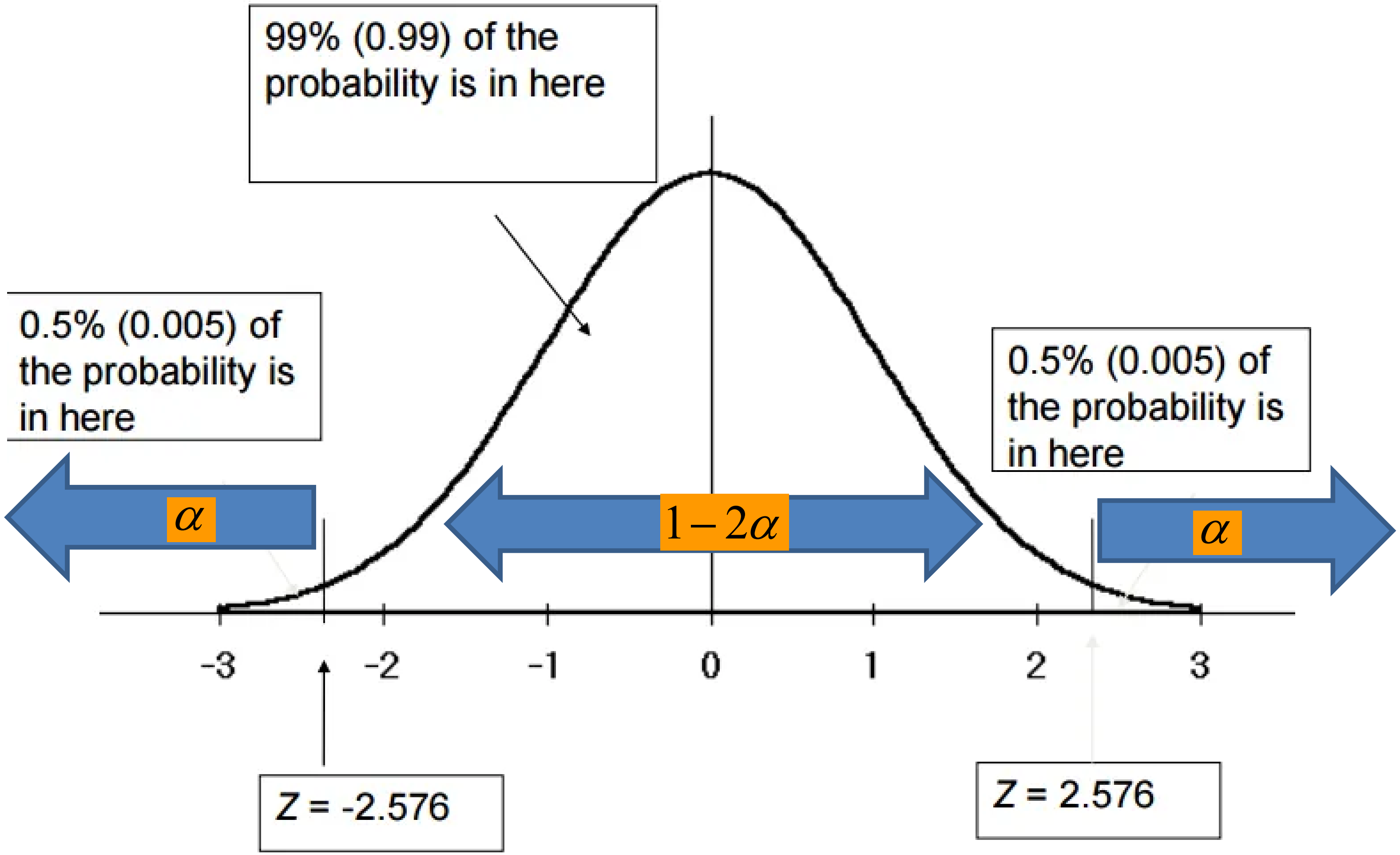
(assuming that x has a normal distribution;-)

standard normal table



z	+0.00	+0.01	+0.02	+0.03	+0.04	+0.05	+0.06	+0.07	+0.08	+0.09
0.0	0.00000	0.00399	0.00798	0.01197	0.01595	0.01994	0.02392	0.02790	0.03188	0.03586
0.1	0.03983	0.04380	0.04776	0.05172	0.05567	0.05962	0.06356	0.06749	0.07142	0.07535
0.2	0.07926	0.08317	0.08706	0.09095	0.09483	0.09871	0.10257	0.10642	0.11026	0.11409
0.3	0.11791	0.12172	0.12552	0.12930	0.13307	0.13683	0.14058	0.14431	0.14803	0.15173
0.4	0.15542	0.15910	0.16276	0.16640	0.17003	0.17364	0.17724	0.18082	0.18439	0.18793
0.5	0.19146	0.19497	0.19847	0.20194	0.20540	0.20884	0.21226	0.21566	0.21904	0.22240
0.6	0.22575	0.22907	0.23237	0.23565	0.23891	0.24215	0.24537	0.24857	0.25175	0.25490
0.7	0.25804	0.26115	0.26424	0.26730	0.27035	0.27337	0.27637	0.27935	0.28230	0.28524
0.8	0.28814	0.29103	0.29389	0.29673	0.29955	0.30234	0.30511	0.30785	0.31057	0.31327
0.9	0.31594	0.31859	0.32121	0.32381	0.32639	0.32894	0.33147	0.33398	0.33646	0.33891
1.0	0.34134	0.34375	0.34614	0.34849	0.35083	0.35314	0.35543	0.35769	0.35993	0.36214
1.1	0.36433	0.36650	0.36864	0.37076	0.37286	0.37493	0.37698	0.37900	0.38100	0.38298

Confidence interval: 1-2α



Summary

1.- Bayes theorem

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3.- Estimation of reliability parameters from tests

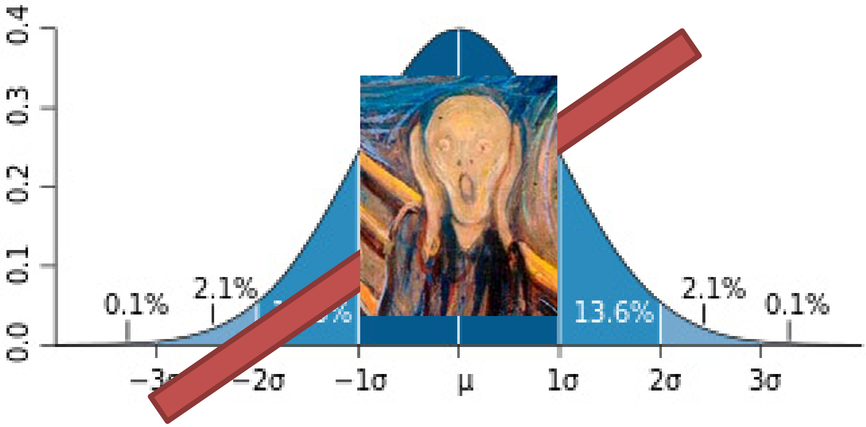
4.- Confidence limits of parameters:

Chi-square distribution

Problem

failure ratio does not follows a normal distribution!!!

$$\hat{\lambda} = \frac{r}{T}$$



Solution

... but 2λT does follow a given distribution. Let me proudly introduce you the chi-square distribution

Chi-Squared distribution

Do you remember the normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(-\frac{1}{2} \left(\frac{x-\mu}{\sigma_x} \right)^2 \right) \Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2 \right)$$

We minimize Z^2 in order to find μ... the following question arises:

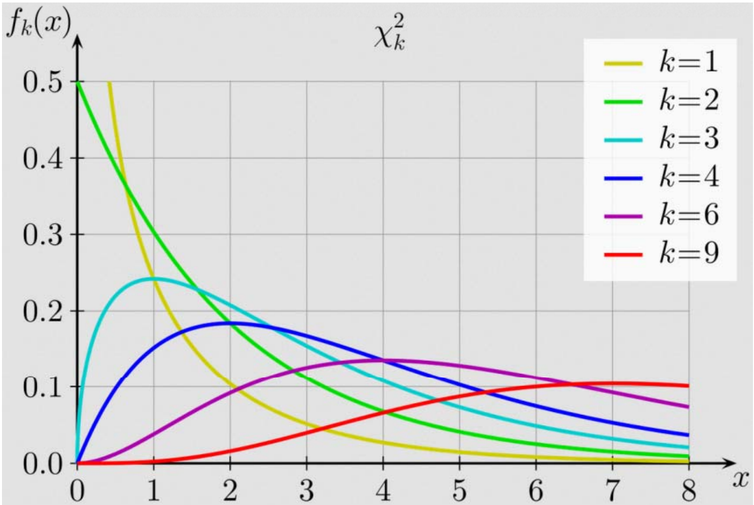
What is the PDF for Z^2 itself?

Chi-Squared distribution

If X_i follow a standard normal distribution then $\chi^2 = \underbrace{X_1^2 + X_2^2 + X_3^2 + \ldots + X_k^2}$

follow a so called χ^2 distribution with **k degrees of freedom** defined as:

$$f(x) = A \cdot x^{k/2-1} e^{-x/2} \qquad A = \frac{1}{2^{k/2}\Gamma\left(k/2\right)}$$



$$\mu = k;\ \mathrm{var} = 2k;\ \max = k-2$$

If k→∞ then $\dfrac{\chi^2 - k}{\sqrt{2k}}$ follows a standard normal distribution , i.e. χ^2 follows N(k,√2k)

$2\lambda T$ is χ^2 distributed with **k** degrees of freedom

Type I (fixed t_0) k=2r+2

Type II (fixed r) k=2r

Let's now calculate the confidence limits:

ONE SIDED (lower limit)

Type I (fixed t_0)
k=2r+2

$$P\left[2\lambda T \le \chi^2_\alpha\left(2+2r\right)\right] = \alpha$$

$$P\left[\lambda \le \frac{\chi^2_\alpha\left(2+2r\right)}{2T}\right] = \alpha$$

$$\theta = \frac{1}{\lambda}$$

$$P\left[\theta \le \frac{2T}{\chi^2_\alpha\left(2+2r\right)}\right] = \alpha$$

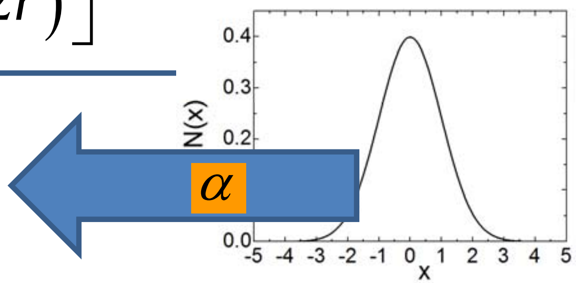
Type I (fixed r
k=2r

$$P\left[2\lambda T \le \chi^2_\alpha\left(2r\right)\right] = \alpha$$

$$P\left[\lambda \le \frac{\chi^2_\alpha\left(2r\right)}{2T}\right] = \alpha$$

$$\theta = \frac{1}{\lambda}$$

$$P\left[\theta \le \frac{2T}{\chi^2_\alpha\left(2r\right)}\right] = \alpha$$



$2\lambda T$ is χ^2 distributed with **k** degrees of freedom

Type I (fixed t_0)
k=2r+2

Type II (fixed r)
k=2r

Let's now calculate the confidence limits:

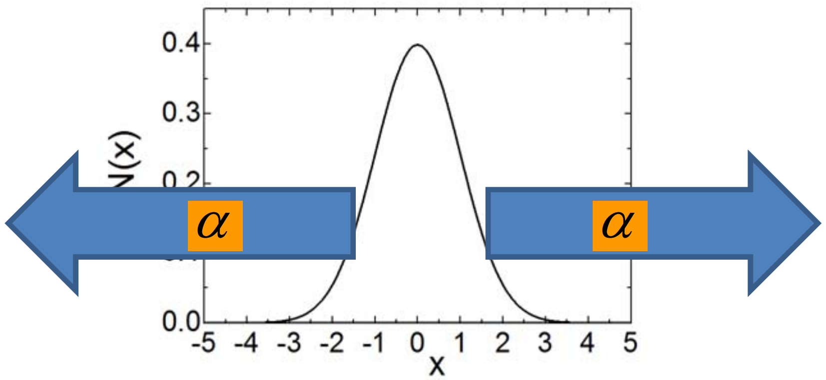
TWO SIDED (lower and upper limit)

Type I (fixed t_0)
k=2r+2

$$P\left[\frac{2T}{\chi^2_{\frac{\alpha}{2}}(2+2r)}\theta \leq \frac{2T}{\chi^2_{1-\frac{\alpha}{2}}(2+2r)}\right]=\alpha$$

Type I (fixed r
k=2r

$$P\left[\frac{2T}{\chi^2_{\frac{\alpha}{2}}(2r)}\theta \leq \frac{2T}{\chi^2_{1-\frac{\alpha}{2}}(2r)}\right]=\alpha$$



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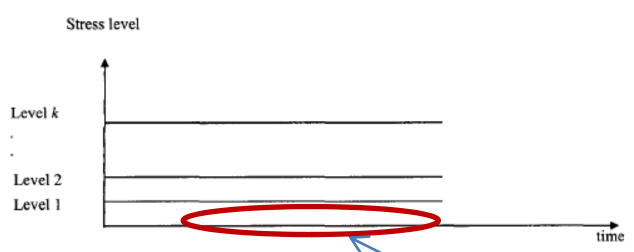
Accelerated Life Tests (ALTs)

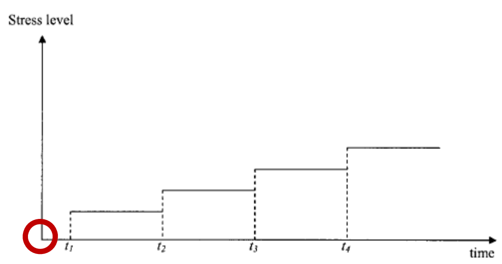
Fortunatelly enough: we cannot wait until the "units" fail...
... so we stress them with voltage, pressure, load, vibration ➔ **stressors S_1, S_2, S_3...**

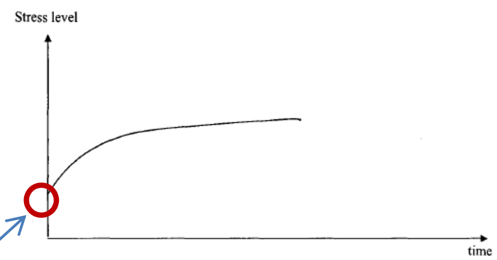
And each stressor has different levels of actuation, being $S_j^{(0)}$ normal stress

in a general case $S_j^{(0)} < S_j^{(1)} < S_j^{(2)} < S_j^{(3)} \ldots S_j^{(k)}$ for stressor j=1,2,... m and levels until k

<u>Design I</u>	**<u>Design II</u>**	**<u>Design III</u>**
censoring type II nj units are randomly exposed to stress S	no censoring nj units are exposed to increasing stress steps S	no censoring nj units are exposed to increasing stress S(t)







The idea is to extrapolate to $S_j^{(0)}$

Accelerated Life Tests (ALTs)

Parametric models

Let's quantify our stress accelerated life tests...

Now our functions are affected by stress, threfore:

Lifetime distribution function (CDF)	$F(t;s) = P(T \leq t;s)$
Associated PDF	$f(t;s) = P(t \leq T \leq t + dt;s)$
Survival function	$R(t;s) = 1 - F(t;s)$
Failure rate	$\lambda(t;s) = \dfrac{f(t;s)}{R(t:s)}$

Accelerated Life Tests (ALTs)

Parametric models

Exponential distribution under design I:

Functional proposals for the relation between stress and failure ratio:

$$\lambda(s) = \frac{1}{c} s^{a}$$

Power rule
- Dielectric breakdown of capacitors
- Fatigue testing

$$\lambda(s) = c e^{-b/s}$$

Arrhenius model
- Themal aging
- Semiconductor materials

$$\lambda(s) = c \cdot s^{-b/s}$$

Eyring model
- constant thermal stress

Accelerated Life Tests (ALTs)

Parametric models for Design I

Design I

censoring type II, nj units are randomly exposed to stress S

Let's calculate the values of a, c and λ

$S^{(j)}$: is the stress level
n_j : number of units tested at stress level j
r_j : number of units failed under stress j
T_{j1} , T_{j2} , ... T_{jnj} = lifetimes of the n_j tested units

The total time in test is $T_j = \sum_{i=1}^{r_j} T_{ji} + \left(n_j - r_j\right) T_{jr_j}$

$$f(T_j) = \frac{dZ_j}{dT_j} A \cdot T_j^{k/2-1} e^{-T_j/2} = 2\lambda$$

As we have seen, Z follows a chi-square distribution with $2r_j$ degrees of freedom

$$Z_j = 2\lambda\left(S^{(j)}\right) T_j$$

$$f(Z_j) = A \cdot Z_j^{k/2-1} e^{-Z_j/2}$$

We change variables from Z to T

$$f(Z) = \frac{dZ}{dT} f(T) = 2\lambda f(T)$$

$$f(T_j) = A \cdot \lambda \left(S^{(j)}\right)^{r^j} T_j^{r_j - 1} e^{-\lambda S^{(j)} T_j}$$

The likelihood thus read as:

$$L = f(T_1,\dots T_k) = \prod_{j=1}^{k} A\cdot\lambda\left(S^{(j)}\right)^{r^j} T_j^{\,r_j-1} e^{-\lambda S^{(j)} T_j}$$

Assuming an exponential relationship

$$\lambda\left(S^{(j)}\right) = \frac{1}{c}\left(\frac{S^{(j)}}{S}\right)$$

$$S = \left[\sqrt{\prod_{j=1}^{k} S^{(j)}}\right]^{r_j \Big/ \sum_{i=1}^{k} r_j}$$

We can now calculate the likelihood, the logarithm and minimize... obtaining

expected value for a

$$\sum_{j=1}^{k}\left[\frac{S^{(j)}}{S}\right]^{\hat{a}} \ln\left[\frac{S^{(j)}}{S}\right] = 0$$

expected value for c

$$c = \frac{1}{\sum_{j=1}^{k} r_i} \sum_{j=1}^{k}\left[\frac{S^{(j)}}{S}\right]^{\hat{a}} \cdot t_j = 0$$

expected value for λ

$$\lambda = \frac{1}{c}\sum_{j=1}^{k}\left[\frac{S^{(0)}}{S}\right]^{\hat{a}}$$

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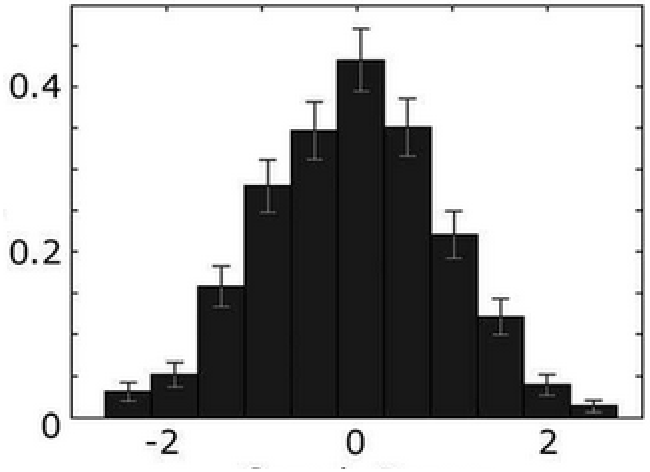
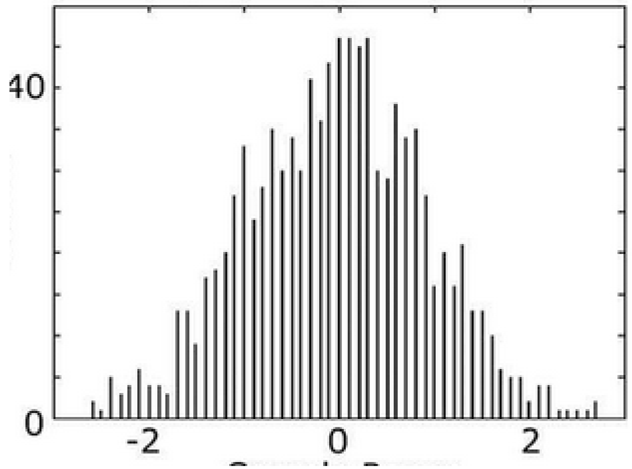
So, we have up to now ASSUMED the distribution f(t), but...
<u>what if the distribution is not known???</u>

Let's assume that there are n units each failing at a time t_i

First we have to distribute them inside boxes: binning
we must therefore decide the size of the binning!





- big bin size
- small error
- few points

- small bin size
- big error
- a lot of points

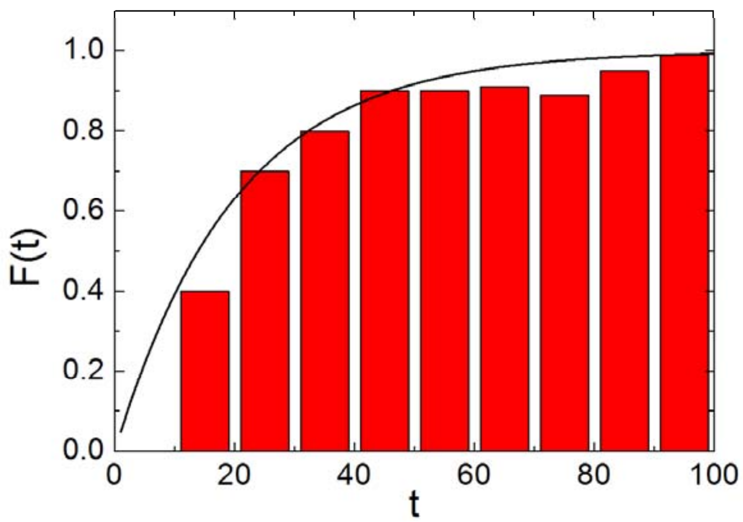
So, we have up to now ASSUMED the distribution f(t), but...
<u>what if the distribution is not known???</u>

Let's assume that there are n units each failing at a time t_i

Now we must see if the obtained curve is a gaussian, or exponential, or weibull...

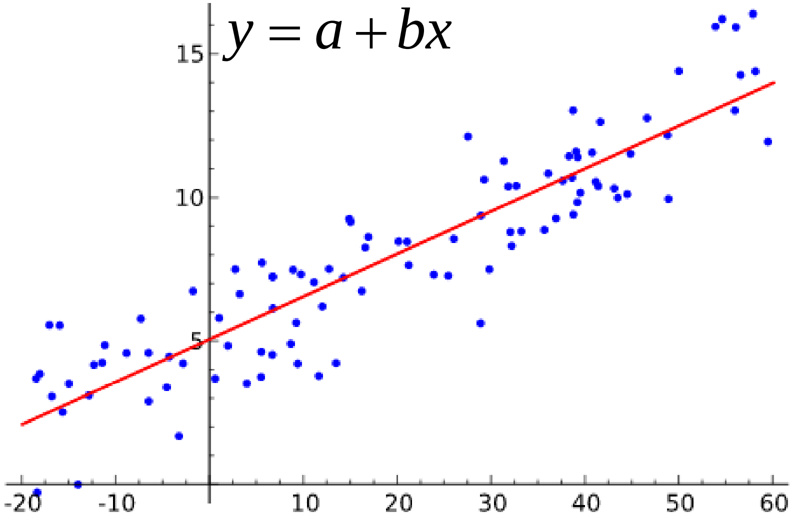
EXAMPLE: exponential. Normally this is done with the cumulative distribution

$$F_T(t) = 1 - e^{-\lambda t}$$



Do the points follow the simple exponential law? or maybe a Weibull $F_T(t) = 1 - e^{-(\lambda t)^{\beta}}$

Fitting data: least squares



Remember the goals:

1.- Parameter estimation
2.- Modelling (Hypothesis testing)

Goal: to minimize the “total distance” of n exp. points $d_1+d_2+\ldots d_n$ to the model

or more seriously.... to minimize the following variabel that follows a χ^2-distribution and therefore D **is normally distributed** around H

$$\chi^2 = \sum_{i=1}^{n} \frac{(D_i - H_i)^2}{H_i}$$

Where D_i are the data pints, and H_i are the model expected values (Hypothesis)

... since for n large a χ^2-distribution for point i tends to a gaussian with $\sigma^2=n=H_i$

$$\chi^2 = \sum_{i=1}^{n} \frac{(D_i - H_i)^2}{H_i} \rightarrow \sum_{i=1}^{n} \frac{(D_i - H_i)^2}{\sigma_i^2}$$

Fitting data: is the model right?

Let's do the question again (more precisse):

•Is the χ^2 arising after minimization
•When assuming that the data are normally distributerd around the model (hypothesis)
•Following a χ^2-distribution of (as it should?)

Rule of thumb (or the joys of the χ^2-distribution)

The χ^2 should follow a χ^2-distribution with n-m degrees of freedom arising for the data (with n points) and the model (with m=2 parameters)

For n data and 2 parameters μ=n-m and therefore the calculated χ^2 :

$$\chi^2 \approx (n-m)$$

For this reason it seems reasonable to define a reduced χ^2 that should be about one

$$\chi^2_{red} = \frac{\chi^2}{n-m} \approx 1$$

... for a good fit

"Hypothesis testing"

Kolmogorov-Smirnov test: WE DO NOT NEED TO DECIDE A BINNING!!!!!

1.- We calculate the cumulative distribution function CDF from the PDF and the model

$$CDF(\chi^2) = \int_0^{\chi^2} f(\chi^2) d\chi^2$$

2.- We look at the maximum distance between the model and the calculated CDF

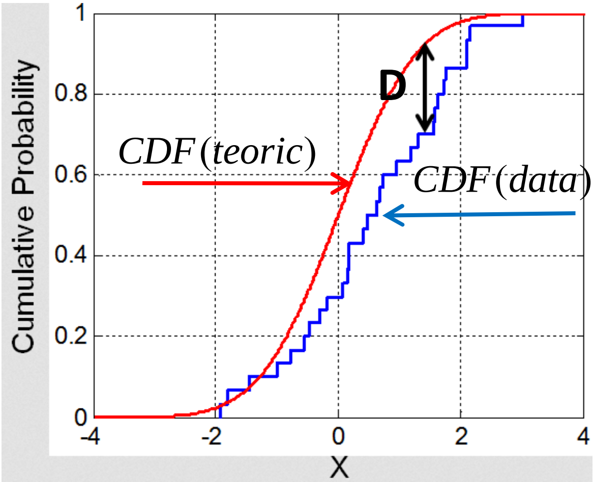
3.- We look at the table for confience limits

confidence limit

number of data

Table 9.2: Critical Values of D_n^α in the Kolmogorov-Smirnov Test [9]

n \ α	0.20	0.10	0.05	0.01
5	0.45	0.51	0.56	0.67
10	0.32	0.37	0.41	0.49
15	0.27	0.30	0.34	0.40
20	0.23	0.26	0.29	0.36
25	0.21	0.24	0.27	0.32
30	0.19	0.22	0.24	0.29
35	0.18	0.20	0.23	0.27
40	0.17	0.19	0.21	0.25
45	0.16	0.18	0.20	0.24
50	0.15	0.17	0.19	0.23
>50	$\frac{1.07}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.63}{\sqrt{n}}$



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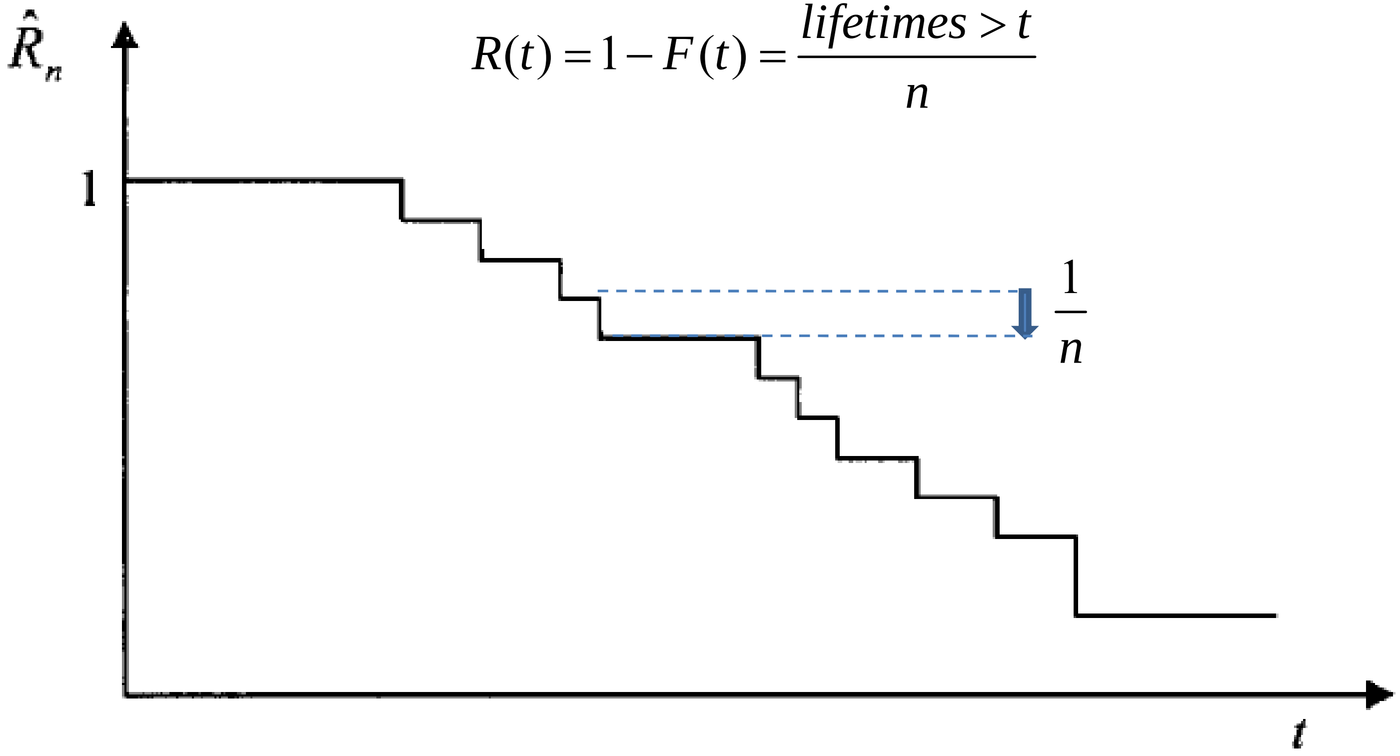
ok... but i don't want to find a model. I want to directly the experimental CDF...

For a non-censored experiment this is quite easy!

Let's do it with the survival function R(t):

t_i : is the lifetime of the unit i=1,2,3,... n

$$R(t) = 1 - F(t) = \frac{lifetimes > t}{n}$$



It must go down from 1 to 0 in n steps, therefore each step MUST be of height 1/n

KAPLAN-MEIER estimator

ok... but i don't want to find a model. I want to directly the experimental CDF...

For a censored experiment is not that easy!

Let's do it with the survival function R(t):

t_i : is the lifetime of the unit i=1,2,3,... n

$(u_i,u_{i+1}]$: is an interval between u_i and u_{i+1} small enough that only a t_i falls into each interval

Let's calculate R(t) at a time t_m

$$R(t_m) = P(T > t_m) = P(T > u_1 \mid T > u_0) \cdot P(T > u_2 \mid T > u_2) \cdot \ldots P(T > t_m \mid T > u_m)$$

$$R(t_m) = P(T > t_m) = \prod_{i=0}^{m} P_i$$

$$\underbrace{P_i = P(T > u_{i+1} \mid T > u_i)}$$

KAPLAN-MEIER estimator

The goal is to calculate P_i

$$R(t_m) = P(T > t_m) = \prod_{i=0}^{m} P_i$$

$$P_i = P(T > u_{i+1} \mid T > u_i)$$

remember:
Intervals $(u_j,u_{j+1}]$ are small enough so that maximum 1 failure occurs

1. If neither failure or censoring occurs P_i=1

2. If censoring occurs, no recording of failures occur, and again P_i=1

3. Imagine that a failure occurs in $(u_j,u_{j+1}]$.
   The number of units at risk before are n_j
   The number of units at risk after are n_j-1

$$P_i = P(T > u_{i+1} \mid T > u_i) = \frac{n_j - 1}{n_n}$$

Therefore the intervals where no failure occur can be misregarded

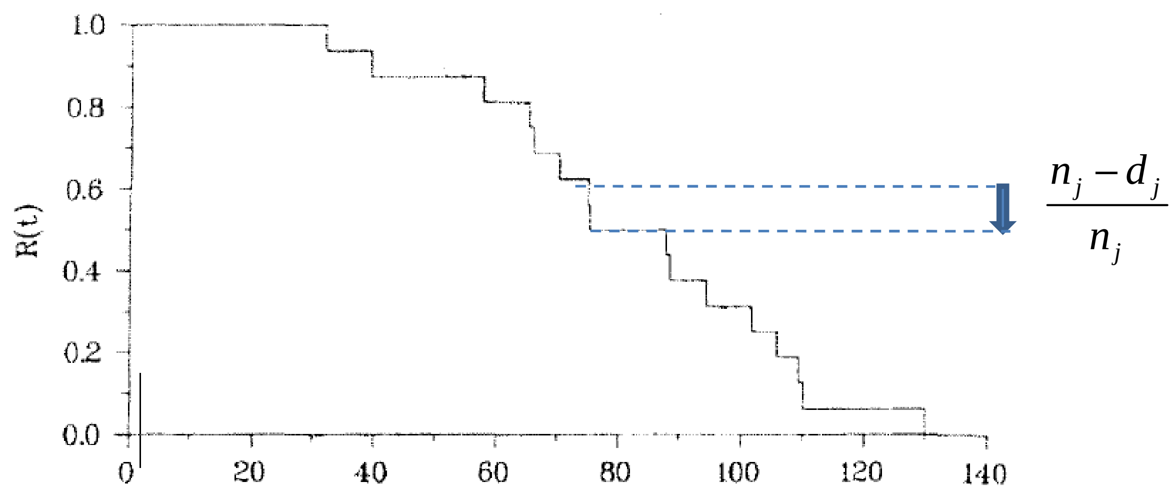
KAPLAN-MEIER estimator

Therefore, R(t) will go down each time a component fail, and R(t) can be represented as:

$$R(t_m) = P(T > t_m) = \prod_{i=0}^{m} P_i = \prod_{j=1}^{n_f} \frac{n_j - 1}{n_j}$$

If more than on unit fails at interval i, then if d_j units fail:

$$R(t_m) = P(T > t_m) = \prod_{i=0}^{m} P_i = \prod_{j=1}^{n_f} \frac{n_j - d_j}{n_j}$$

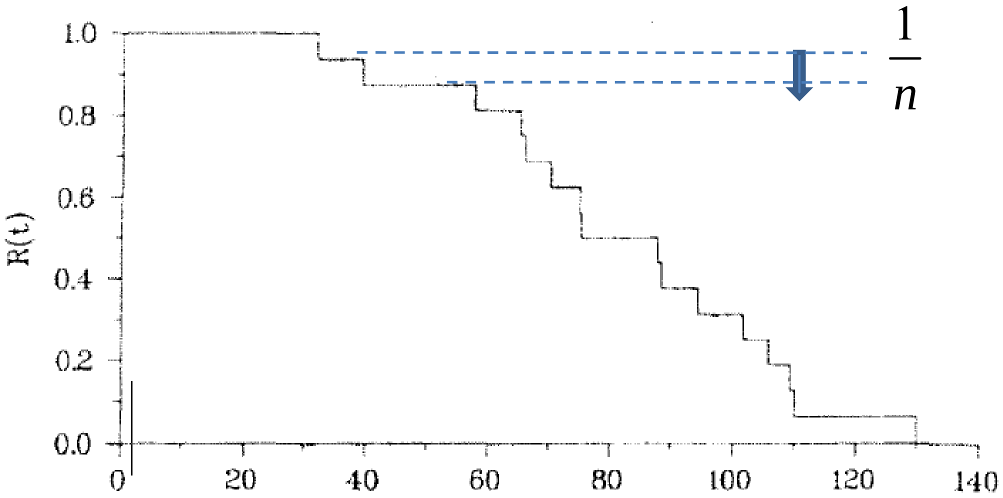


KAPLAN-MEIER estimator

EXAMPLE:
A test is carried out for n=16 units, obtaining the folloring failure times

31.7 ; 39.2 ; 57.5 ; 65.0 ; 65.8 ; 70.0 ; 75.0 ; 75.2 ;
87.5 ; 88.3 ; 94.2 ; 101.7 ; 105.8 ; 109.2 ; 110.0 ; 130.0

Calculate the survival function R(t)



KAPLAN-MEIER estimator

EXAMPLE (same times as before):
A test is carried out for n=16 units, obtaining the folloring failure times

31.7 ; 39.2 ; 57.5 ; 65.8 ; 70.0 ; 105.8 ; 110.0

The rest are censored tests Calculate the survival function R(t)

Rank j	Inverse Rank n-j+1	Ordered Failure and Censoring Times t_j	$\hat{p}_j$	$\hat{R}(t_{(j)})$
0	-	-	1	1.000
1	16	31.7	15/16	0.938
2	15	39.2	14/15	0.875
3	14	57.2	13/14	0.813
4	13	65.0*	1	0.813
5	12	65.8	11/12	0.745
6	11	70.0	10/11	0.677
7	10	75.0*	1	0.677
8	9	75.2*	1	0.677
9	8	87.5*	1	0.677
10	7	88.3*	1	0.677
11	6	94.2*	1	0.677
12	5	101.7*	1	0.677
13	4	105.8	¾	0.508
14	3	109.2*	1	0.508
15	2	110.0	½	0.254
16	1	130.0*	1	0.254

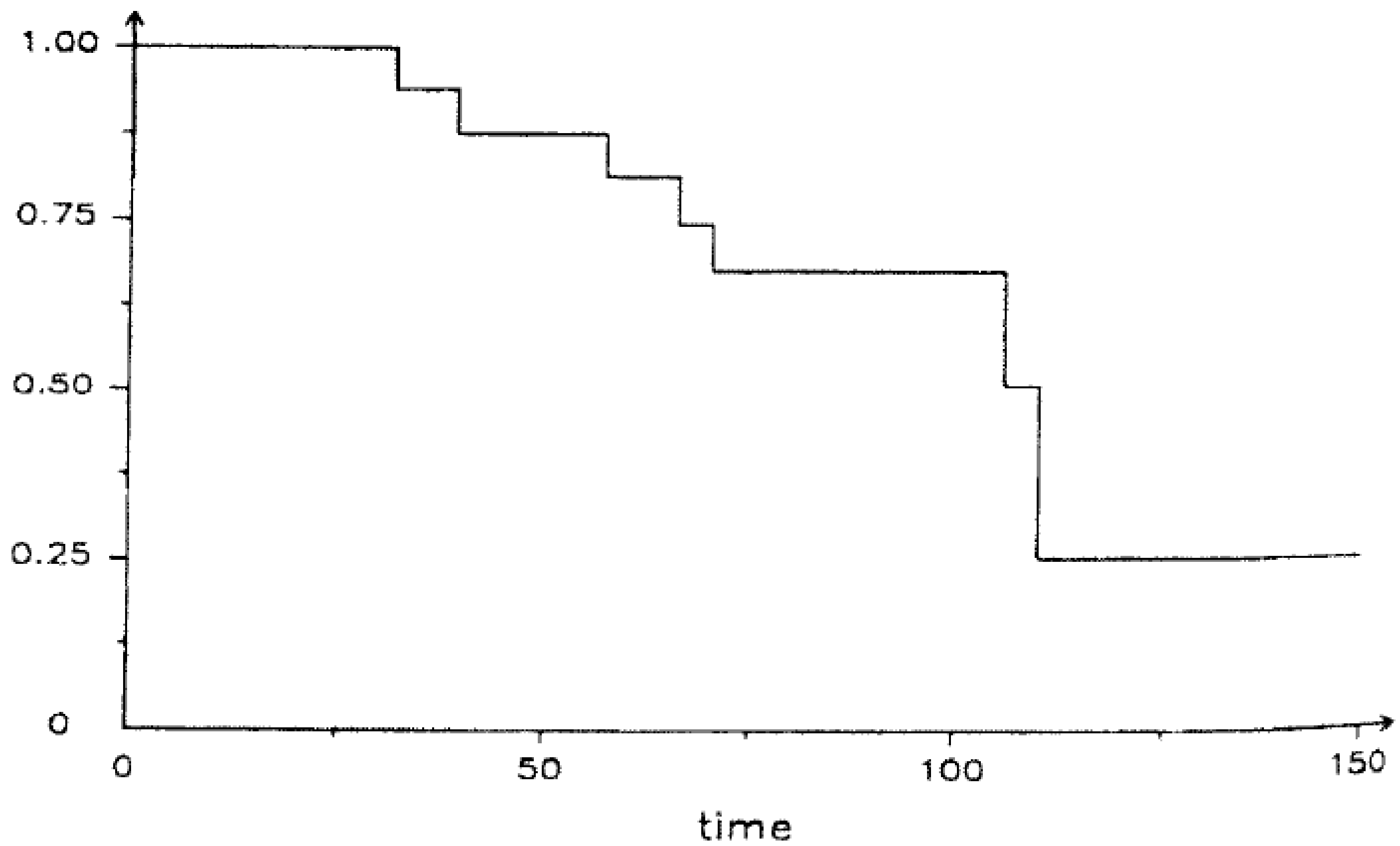
t	$\hat{R}(t)$
$0 \le t < 31.7$	=1
$31.7 \le t < 39.2$	15/16=0.938
$39.2 \le t < 57.5$	15/16·14/15=0.875
$57.5 \le t < 65.8$	15/16·14/15·13/14=0.813
$65.8 \le t < 70.0$	15/16·14/15·13/14·11/12=0.745
$70.0 \le t < 105.8$	15/16·14/15·13/14·11/12·10/11=0.677
$105.8 \le t < 110.0$	15/16·14/15·13/14·11/12·10/11·3/4=0.508
$110.0 \le t$	15/16·14/15·13/14·11/12·10/11·3/4·1/2=0.254

KAPLAN-MEIER estimator

EXAMPLE (same times as before):
A test is carried out for n=16 units, obtaining the folloring failure times

31.7 ; 39.2 ; 57.5 ; 65.8 ; 70.0 ; 105.8 ; 110.0

The rest are censored tests Calculate the survival function R(t)



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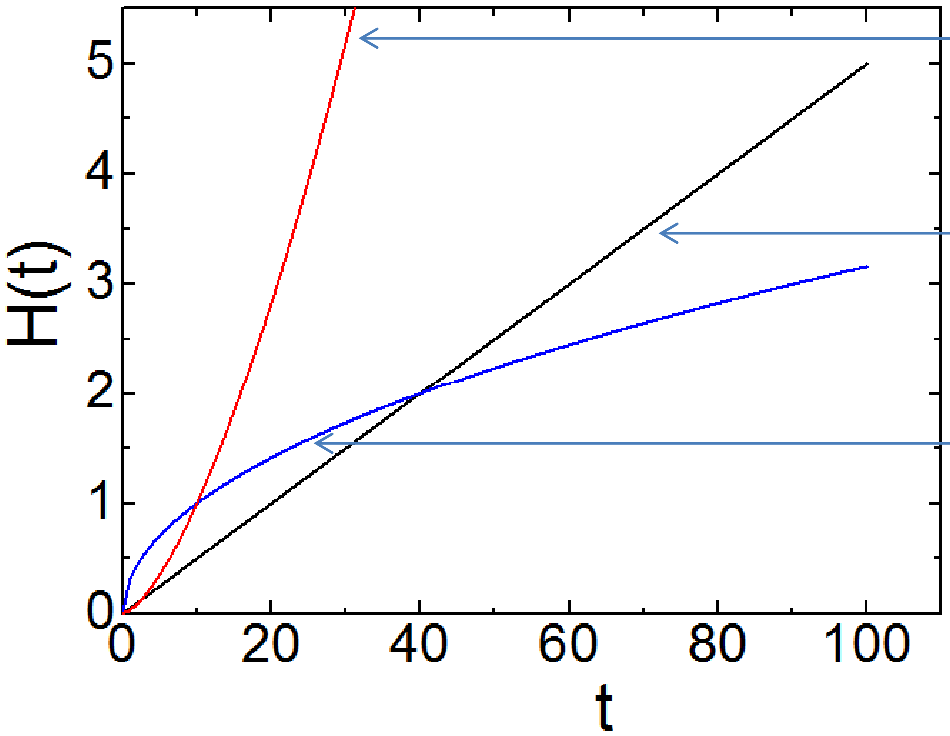
Reliability growth:

Engineering changes cause an increase of reliability

Can be quantified:

- Cumulative number of failures as a function of time
- Failure rate **h(t)=λ(t)** as a function of time
- Mean time between failures (**MTBF**) as a function of time

We need, in any case, a reliability growth model



Duane model (β>0)

$$H(t) = \left(\frac{t}{\tau}\right)^{\beta}$$

Exponential (β=0)

$$H(t) = \lambda \cdot t = \frac{t}{\tau}$$

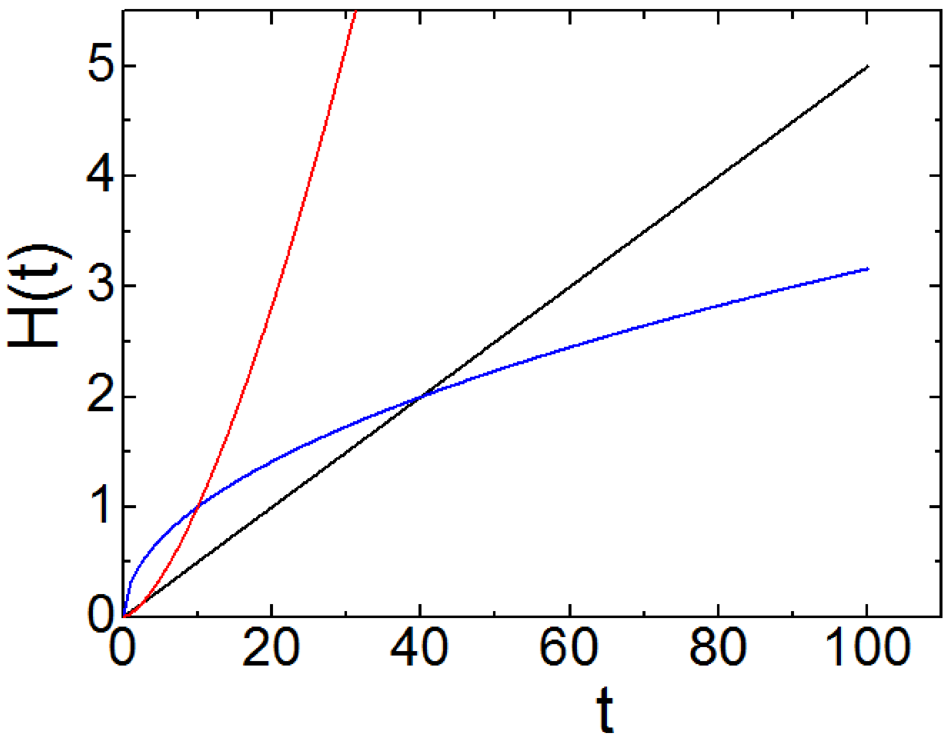
Duane model (β<0)

$$H(t) = \left(\frac{t}{\tau}\right)^{\beta}$$

Let's find the parameters of the Duane model: **Maximum likelihood**

Duane model (β>0)

$$H(t) = \left(\frac{t}{\tau}\right)^{\beta}$$



Duane model (β<0)

$$H(t) = \left(\frac{t}{\tau}\right)^{\beta}$$

Calculating de likelihood, making the logariothm and minimizing:

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{T}{t_i}\right)} \qquad \hat{\alpha} = \frac{T}{n^{1/\beta}}$$

Let's find the parameters of the Duane model: **Least squares**

Example:
Let's calculate the parameters of the Duane analysis for the following data

(1)	(2)	(3)	(4)	(5)	(6)
Month of Operation	Hours of Operation	Cumulative Hours t	No. of failures	Cumulative Number of failures $H(t)$	Cumulative MTBF $t/H'(t)$
1	541	541	3	3	180.3
2	1171	1712	5	8	214.0
3	1939	3651	4	12	304.3
4	2403	6054	1	13	465.7
5	1718	7772	2	15	518.1
6	2206	9978	2	17	586.9
7	1366	11244	3	20	562.2
8	1529	12873	0	20	643.7
9	1449	14322	2	22	651.0
10	1451	15773	2	24	657.2

Let's find the parameters of the Duane model: **Least squares**

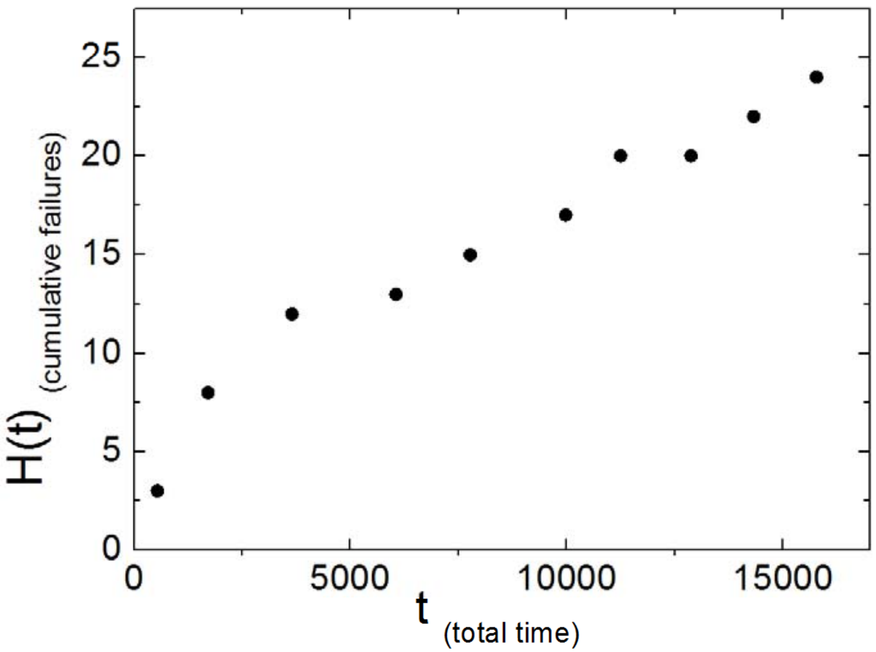
Let's first do the figure:

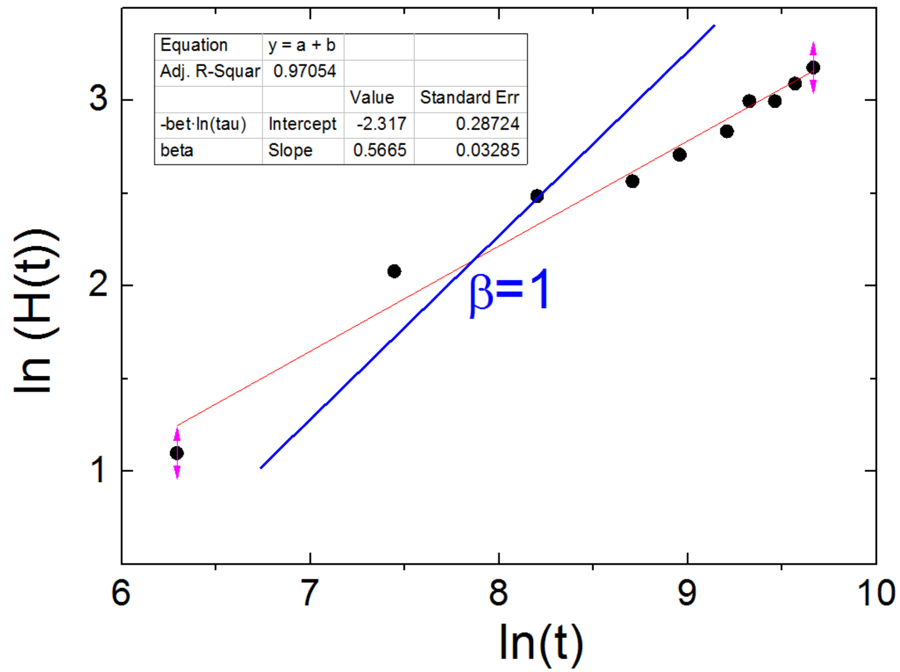
Duane Model

$$H(t) = \left(\frac{t}{\tau}\right)^{\beta}$$

linearized Duane Model

$$\ln\left(H(t)\right) = \beta \ln t - \beta \ln \tau$$





β<1 : reliability growth

Let's now add MONEY

Imagine a system whose failure causes production stoppage:

- Failure costs *Cr* (include production stoppage)
- Complete overhaul costs *Co* (includes lost of production, replacement and labour)

What is the optimum overhaul policy?

The total cost is

$$C(t) = C_r H(t) + C_o$$

C_o ← cost total overhaul

$H(t)$ ↑ # units that fail

C_r ↑ cost to repair

And we just showed the Duane model for H(t)

$$H(t) = \left(\frac{t}{\tau}\right)^{\beta}$$

Therefore , the cost/unit operating time is

$$\gamma(t) = \frac{C(t)}{t} = \frac{C_r\left(\frac{t}{\tau}\right)^{\beta} + C_o}{t}$$

and we want o minimize this cost...

Therefore , the cost/unit operating time is $$\gamma(t) = \frac{C(t)}{t} = \frac{C_r\left(\frac{t}{\tau}\right)^{\beta} + C_o}{t}$$

and we want o minimize this cost...

We solve the equation $$\frac{d\gamma(t)}{dt} = 0$$

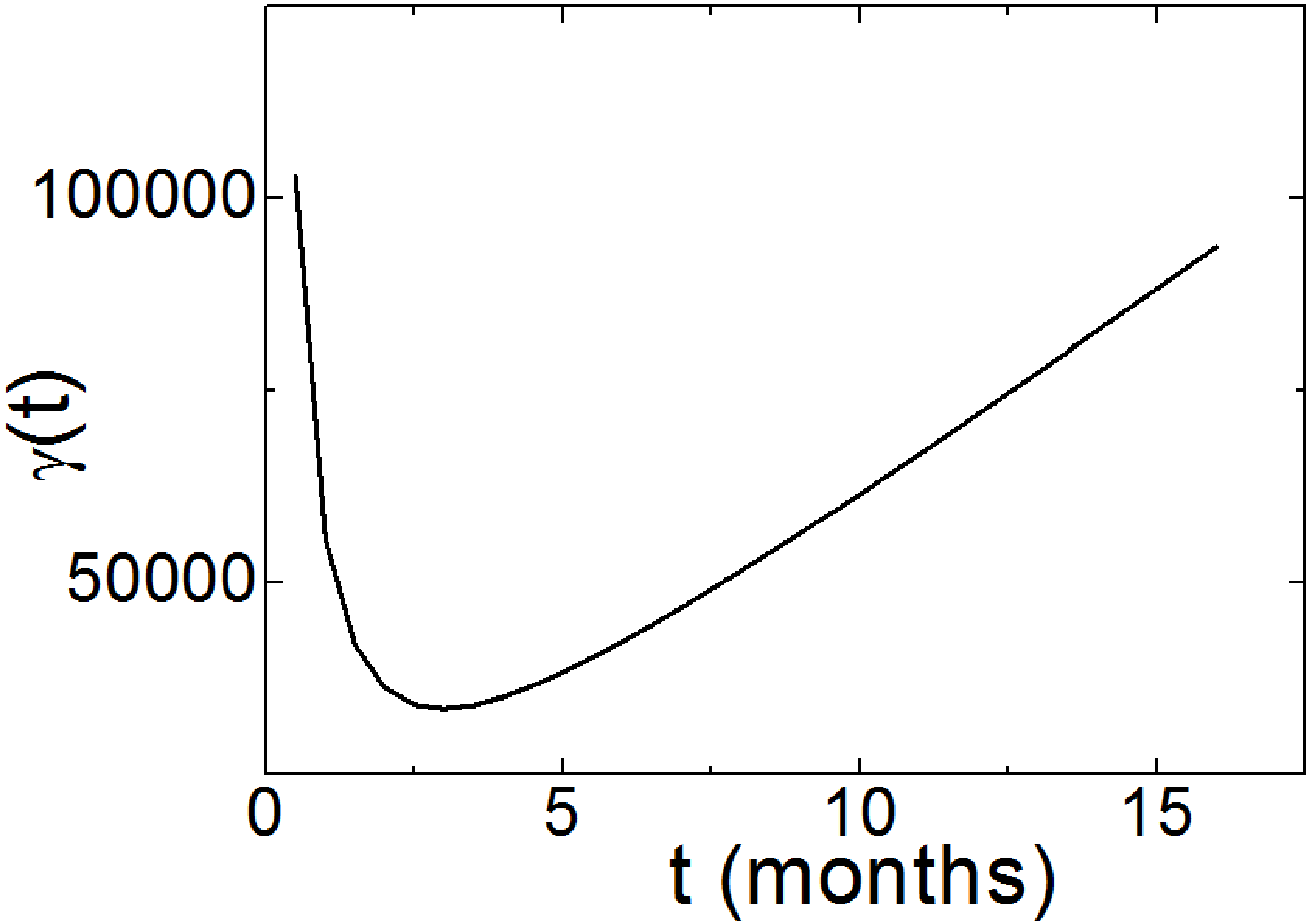
And we obtain that the optimal overhaul time is

$$t^* = \tau\left[\frac{\alpha^{\beta} C_o}{C_r(\beta-1)}\right]^{1/\beta}$$

The overhaul time increases with Co... since it is expensive to do,
and increases with Cr, since if it is expensive is better to cahnge it

And the time is ∞ for β=1 (exponential)... therefore do never change

We can learn more looking at the cost/unit operating time

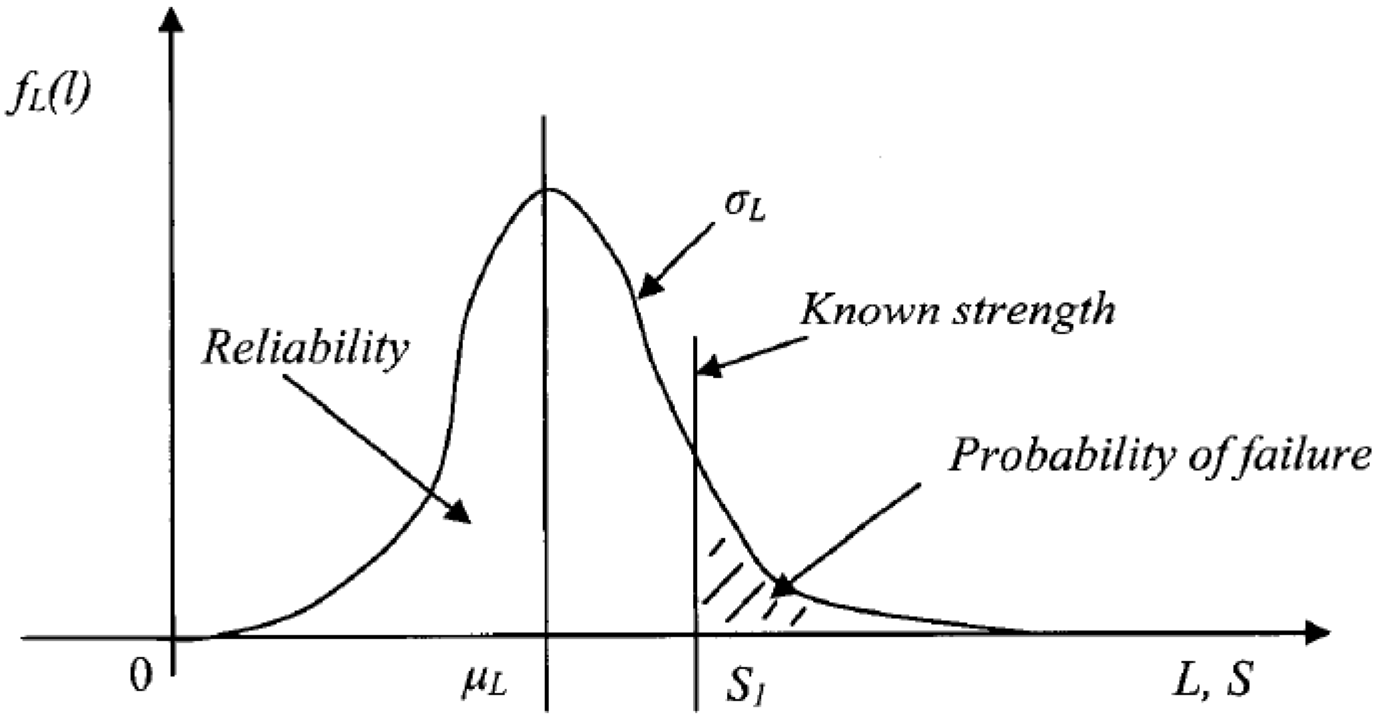


... when in doubt, do it later!

Summary

1.- Bayes theorem
2.- Maximum likelihood method
3.- Estimation of reliability parameters from tests
4.- Confidence limits of parameters
5.- Accelerated life testing
6.- Determination of distribution models
7.- Empirical determination of survivor function
8.- Reliability growth
9.- Strength-stress models

Units fail when the **stress** is to high for their **strength**



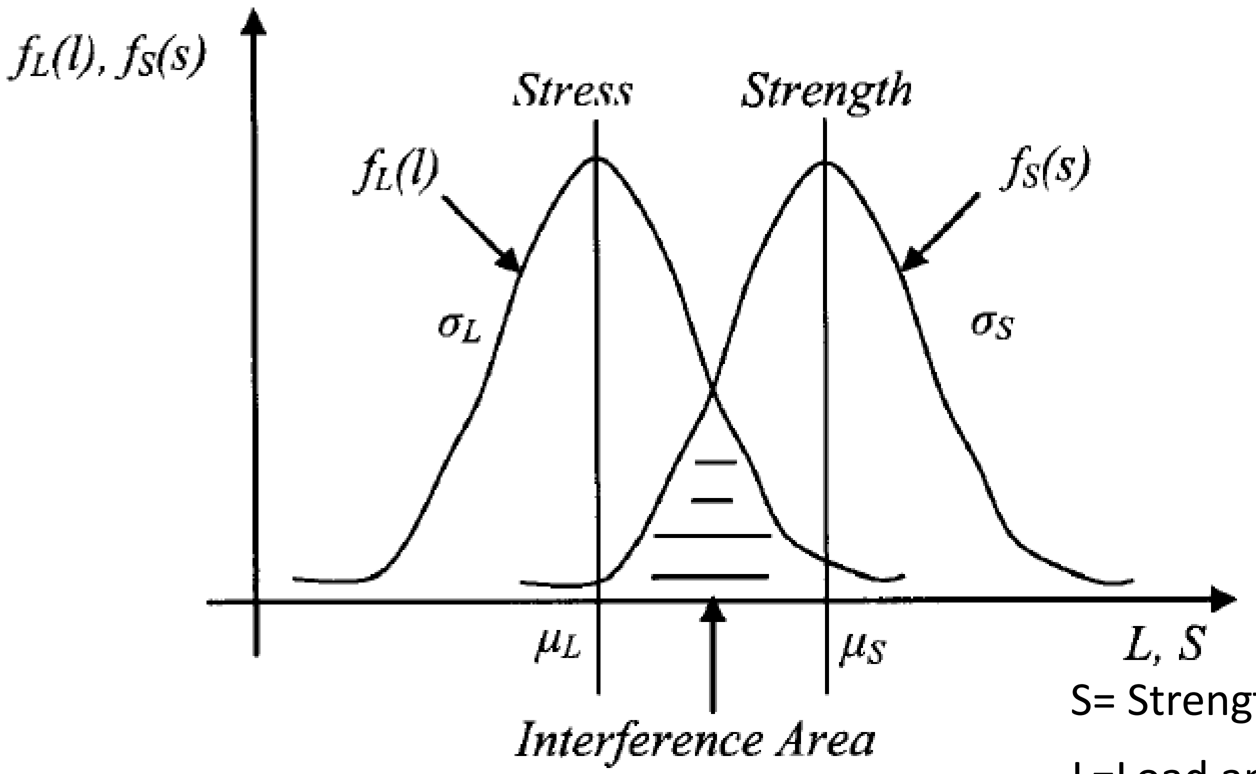
S= Strength of the unit

L=Load apprlied to the unit

Probability of failure:

$$F = P\left[L > S_1\right] = \int_{S_1}^{\infty} f\left(l\right) dl$$

Units fail when the **stress** is to high for their **strength**
but stress might also have a probability distribution. Then

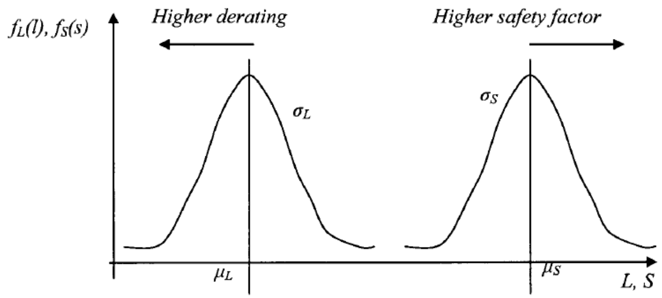


S= Strength of the unit

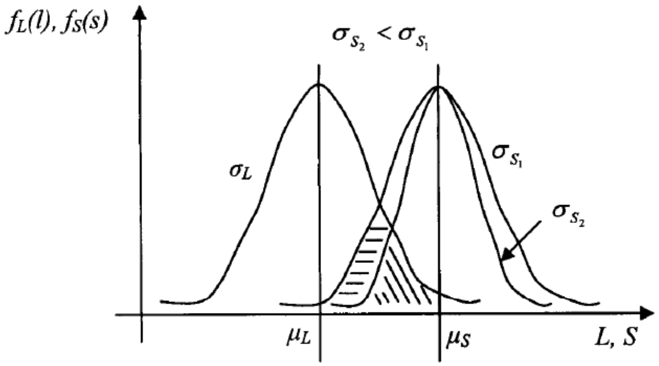
L=Load apprlied to the unit

To henance the reliability there are three option

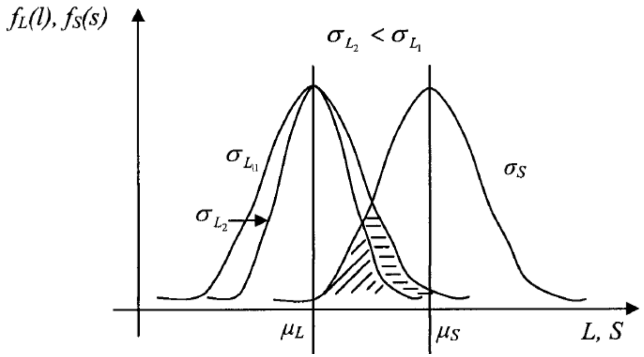
• Shift apart stress and strength distributions



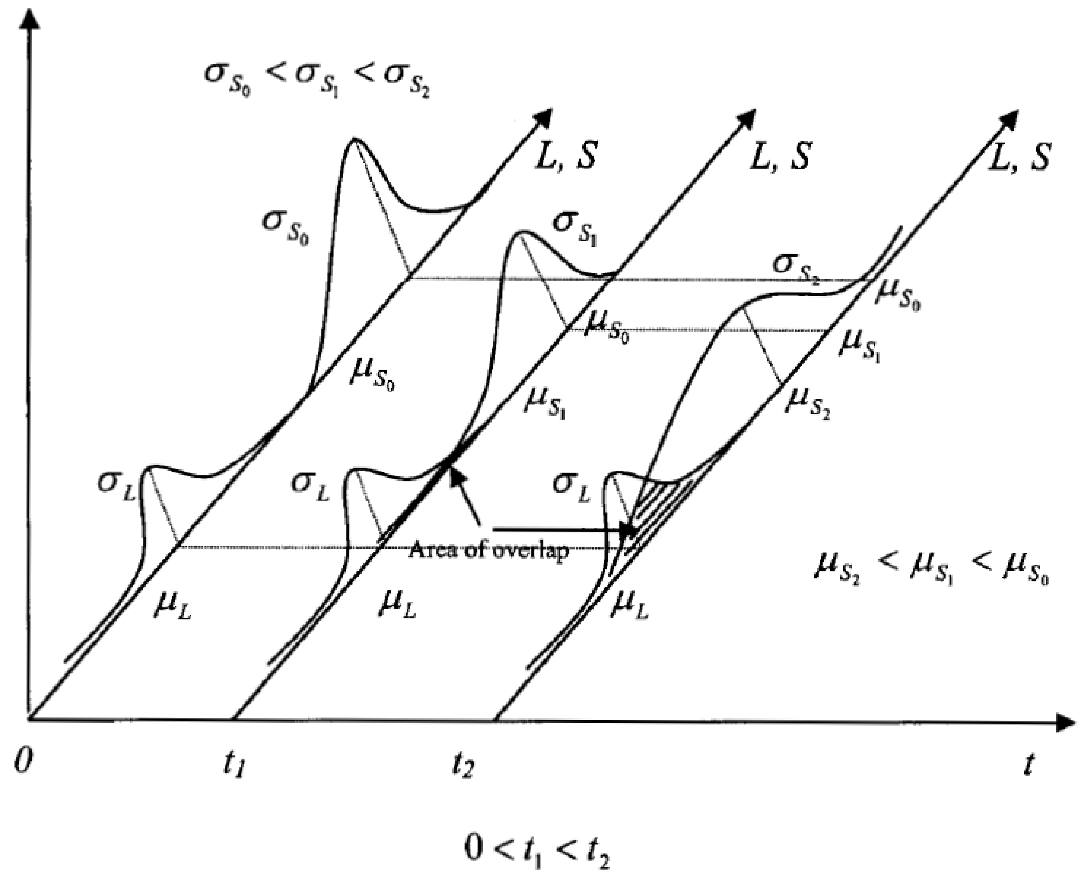
• Reduce the variability of stress



•Reduce the variability of strength (good quality controls!)



The problem is that strength varies over time while stress does not



A good quality

Appendix A: Table of Standard Normal Cumulative Distribution

$$F(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi}e^{-\frac{1}{2}x^2}\,dx$$

ξ	$F(\xi)$
0.00	0.500000
0.01	0.503989
0.02	0.507978
0.03	0.511966
0.04	0.515954
0.05	0.519939
0.06	0.523922
0.07	0.527904
0.08	0.531882
0.09	0.535857
0.10	0.539828
0.11	0.543796
0.12	0.547759
0.13	0.551717
0.14	0.555671
0.15	0.559618
0.16	0.563500
0.17	0.567494
0.18	0.571423
0.19	0.575345
0.20	0.579260
0.21	0.583166
0.22	0.587064
0.23	0.590954
0.24	0.549835
0.25	0.598706
0.26	0.602568
0.27	0.606420
0.28	0.610262
0.29	0.614092

ξ	$F(\xi)$
0.50	0.691463
0.51	0.694975
0.52	0.698468
0.53	0.701944
0.54	0.705401
0.55	0.708840
0.56	0.712260
0.57	0.715661
0.58	0.719043
0.59	0.722405
0.60	0.725747
0.61	0.729069
0.62	0.732371
0.63	0.735653
0.64	0.738914
0.65	0.742154
0.66	0.745374
0.67	0.748572
0.68	0.751748
0.69	0.754903
0.70	0.758036
0.71	0.761148
0.72	0.764238
0.73	0.767305
0.74	0.770350
0.75	0.773373
0.76	0.776373
0.77	0.779350
0.78	0.782305
0.79	0.785236

ξ	$F(\xi)$
1.00	0.841345
1.01	0.843752
1.02	0.846136
1.03	0.848495
1.04	0.850830
1.05	0.853141
1.06	0.855428
1.07	0.857690
1.08	0.859929
1.09	0.862143
1.10	0.864334
1.11	0.866500
1.12	0.868643
1.13	0.870762
1.14	0.872857
1.15	0.874928
1.16	0.876976
1.17	0.878999
1.18	0.881000
1.19	0.882977
1.20	0.884930
1.21	0.886860
1.22	0.888767
1.23	0.890651
1.24	0.892512
1.25	0.894350
1.26	0.896165
1.27	0.897958
1.28	0.899727
1.29	0.901475

ξ	$F(\xi)$
0.30	0.617912
0.31	0.621720
0.32	0.623517
0.33	0.629301
0.34	0.633072
0.35	0.636831
0.36	0.640576
0.37	0.644309
0.38	0.648027
0.39	0.651732
0.40	0.655422
0.41	0.659097
0.42	0.662757
0.43	0.666402
0.44	0.670032
0.45	0.673645
0.46	0.677242
0.47	0.680823
0.48	0.684387
0.49	0.687933
1.50	0.933193
1.51	0.934478
1.52	0.935744
1.53	0.936992
1.54	0.938220
1.55	0.939429
1.56	0.940620
1.57	0.941792
1.58	0.942947
1.59	0.944083
1.60	0.945201
1.61	0.946301
1.62	0.947384
1.63	0.948449
1.64	0.949497

ξ	$F(\xi)$
0.80	0.788145
0.81	0.791030
0.82	0.793892
0.83	0.796731
0.84	0.799546
0.85	0.802337
0.86	0.805105
0.87	0.807850
0.88	0.810570
0.89	0.813267
0.90	0.815940
0.91	0.818589
0.92	0.821214
0.93	0.823815
0.94	0.826391
0.95	0.828944
0.96	0.831473
0.97	0.833977
0.98	0.836457
0.99	0.838913
2.00	0.977250
2.01	0.977784
2.02	0.978308
2.03	0.978822
2.04	0.979325
2.05	0.979818
2.06	0.980301
2.07	0.980774
2.08	0.981237
2.09	0.981691
2.10	0.982136
2.11	0.982571
2.12	0.982997
2.13	0.983414
2.14	0.983823

ξ	$F(\xi)$
1.30	0.903199
1.31	0.904902
1.32	0.906583
1.33	0.908241
1.34	0.909877
1.35	0.911492
1.36	0.913085
1.37	0.914656
1.38	0.916207
1.39	0.917735
1.40	0.919243
1.41	0.920730
1.42	0.922196
1.43	0.923641
1.44	0.925066
1.45	0.926471
1.46	0.927855
1.47	0.929219
1.48	0.930563
1.49	0.931888
2.50	0.993790
2.51	0.993963
2.52	0.994132
2.53	0.994267
2.54	0.994457
2.55	0.994614
2.56	0.994766
2.57	0.994915
2.58	0.995060
2.59	0.995201
2.60	0.995339
2.61	0.995473
2.62	0.995604
2.63	0.995731
2.64	0.995855

ξ	$F(\xi)$
1.65	0.950529
1.66	0.951543
1.67	0.952540
1.68	0.953521
1.69	0.954486
1.70	0.955435
1.71	0.956367
1.72	0.957284
1.73	0.958185
1.74	0.959071
1.75	0.959941
1.76	0.960796
1.77	0.961636
1.78	0.962426
1.79	0.963273
1.80	0.964070
1.81	0.964852
1.82	0.965621
1.83	0.966375
1.84	0.967116
1.85	0.967843
1.86	0.968557
1.87	0.969258
1.88	0.969946
1.89	0.970621
1.90	0.971284
1.91	0.971933
1.92	0.972571
1.93	0.973197
1.94	0.973810
1.95	0.974412
1.96	0.975002
1.97	0.975581
1.98	0.976148
1.99	0.976705

ξ	$F(\xi)$
2.15	0.984223
2.16	0.984614
2.17	0.984997
2.18	0.985371
2.19	0.985738
2.20	0.986097
2.21	0.986447
2.22	0.986791
2.23	0.987126
2.24	0.987455
2.25	0.987776
2.26	0.988089
2.27	0.988396
2.28	0.988696
2.29	0.988989
2.30	0.989276
2.31	0.989556
2.32	0.989830
2.33	0.990097
2.34	0.990358
2.35	0.990613
2.36	0.990863
2.37	0.991106
2.38	0.991344
2.39	0.991576
2.40	0.991802
2.41	0.992024
2.42	0.992240
2.43	0.992451
2.44	0.992656
2.45	0.992857
2.46	0.993053
2.47	0.993244
2.48	0.993431
2.49	0.993613

ξ	$F(\xi)$
2.65	0.995975
2.66	0.996093
2.67	0.996207
2.68	0.996319
2.69	0.996427
2.70	0.996533
2.71	0.996636
2.72	0.996736
2.73	0.996833
2.74	0.996928
2.75	0.997020
2.76	0.997110
2.77	0.997197
2.78	0.997282
2.79	0.997365
2.80	0.997445
2.81	0.997523
2.82	0.997599
2.83	0.997673
2.84	0.997744
2.85	0.997814
2.86	0.997882
2.87	0.997948
2.88	0.998012
2.89	0.998074
2.90	0.998134
2.91	0.998193
2.92	0.998250
2.93	0.998305
2.94	0.998359
2.95	0.998411
2.96	0.998462
2.97	0.998511
2.98	0.998559
2.99	0.998605

ξ	$F(\xi)$
3.00	0.998630
3.01	0.998694
3.02	0.998736
3.03	0.998777
3.04	0.998817
3.05	0.998856
3.06	0.998893
3.07	0.998930
3.08	0.998965
3.09	0.998999
3.10	0.999032
3.11	0.999065
3.12	0.999096
3.13	0.999126
3.14	0.999155
3.15	0.992184
3.16	0.999119
3.17	0.999238
3.18	0.999264
3.19	0.999289
3.20	0.999313
3.21	0.999336
3.22	0.999359
3.23	0.999381
3.24	0.999402
3.25	0.999423
3.26	0.999443
3.27	0.999462
3.28	0.999481
3.29	0.999499
3.30	0.999516
3.31	0.999533
3.32	0.999550
3.33	0.999566
3.34	0.999581

ξ	$F(\xi)$
3.50	0.999767
3.51	0.999776
3.52	0.999784
3.53	0.999792
3.54	0.999800
3.55	0.999807
3.56	0.999815
3.57	0.999821
3.58	0.999828
3.59	0.999835
3.60	0.999841
3.61	0.999847
3.62	0.999853
3.63	0.999858
3.64	0.999864
3.65	0.999869
3.66	0.999874
3.67	0.999879
3.68	0.999883
3.69	0.999888
3.70	0.999892
3.71	0.999806
3.72	0.999900
3.73	0.999904
3.74	0.999908
3.75	0.999912
3.76	0.999915
3.77	0.999918
3.78	0.999922
3.79	0.999925
3.80	0.999928
3.81	0.999931
3.82	0.999933
3.83	0.999936
3.84	0.999938

ξ	$1-F(\xi)$
4.00	0.316712E-04
4.05	0.256088E-04
4.10	0.206575E-04
4.15	0.166238E-04
4.20	0.133458E-04
4.25	0.106883E-04
4.30	0.853906E-05
4.35	0.680688E-05
4.40	0.541234E-05
4.45	0.429351E-05
4.50	0.339767E-05
4.55	0.268230E-05
4.60	0.211245E-05
4.65	0.165968E-05
4.70	0.130081E-05
4.75	0.101708E-05
4.80	0.793328E-06
4.85	0.617307E-06
4.90	0.479183E-06
4.95	0.371067E-06
5.00	0.286652E-06
5.10	0.169827E-06
5.20	0.996443E-07
5.30	0.579013E-07
5.40	0.333204E-07
5.50	0.189896E-07
5.60	0.107176E-07
5.70	0.599037E-08
5.80	0.331575E-08
5.90	0.181751E-08
6.00	0.986588E-09
6.10	0.530343E-09
6.20	0.282316E-09
6.30	0.148823E-09
6.40	0.77688 E-10

ξ	$F(\xi)$
3.35	0.999596
3.36	0.999610
3.37	0.999624
3.38	0.999637
3.39	0.999650
3.40	0.999663
3.41	0.999675
3.42	0.999687
3.43	0.999698
3.44	0.999709
3.45	0.999720
3.46	0.999730
3.47	0.999740
3.48	0.999749
3.49	0.999758

ξ	$F(\xi)$
3.85	0.999941
3.86	0.999943
3.87	0.999946
3.88	0.999948
3.89	0.999950
3.90	0.999952
3.91	0.999954
3.92	0.999956
3.93	0.999958
3.94	0.999959
3.95	0.999961
3.96	0.999963
3.97	0.999964
3.98	0.999966
3.99	0.999967

ξ	$1-F(\xi)$
6.50	0.40160 E-10
6.60	0.20558 E-10
6.70	0.10421 E-10
6.80	0.5231 E-11
6.90	0.260 E-11
7.00	0.128 E-11
7.10	0.624 E-12
7.20	0.301 E-12
7.30	0.144 E-12
7.40	0.68 E-13
7.50	0.32 E-13
7.60	0.15 E-13
7.70	0.70 E-14
7.80	0.30 E-14
7.90	0.15 E-14

Appendix B: Table of Chi-Square Cumulative Distribution

$\chi_{\alpha}^{2}(f)$

$f \backslash \alpha$	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	—	—	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169

$$f(x) = A \cdot x^{k/2-1} e^{-x/2}$$

$$A = \frac{1}{2^{k/2}\,\Gamma\left(k/2\right)}$$