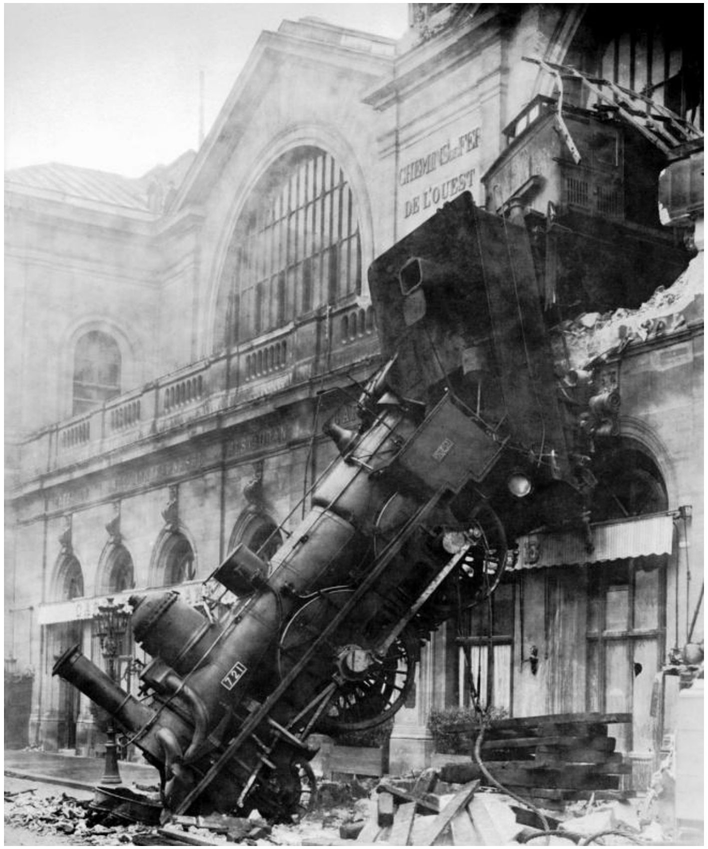
Unit 2



Estimation of parameters

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Summary

1.- Bayes theorem
2.- Maximum likelihood method
3.- Estimation of reliability parameters from tests
4.- Confidence limits of parameters
5.- Accelerated life testing
6.- Determination of distribution models
7.- Empirical determination of survivor function
8.- Reliability growth
9.- Strength-stress models

BAYES THEOREM

From the “multiplication” of probability:

$$prob\left(A,B\right) = prob(A)\cdot prob(B\,|\,A)$$

$$prob\left(A,B\right) = prob(B)\cdot prob(A\,|\,B)$$

dividing the two eqations we get:

Bayes theorem

$$prob\left(A\,|\,B\right) = \frac{prob\left(B\,|\,A\right) prob(A)}{prob(B)}$$

That help us to reverse probabilities...

Bayes theorem

$$prob(H|D) = \frac{prob(D|H)\,prob(H)}{prob(D)}$$

Posterior prob(H|D):
What you want to know is the probability that your hypothesis is true given the data

Likelihood (or L) prob(D|H):
What you know is yout hypothesis. And therefore you can calculate the probability that your data “gathers” around your hypothesis

Prior prob(H):
You might want to include any prioir information about your hypothesis

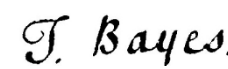
Evidence (E) prob(D):
Is “simply” a normalization factor...

Bayes theorem

Likelihood
Probability that our data describes the hypothesis

Prior
Our forehand knowledge

$$P(H \mid D) = \frac{P(D \mid H) \cdot P(H)}{P(D)}$$



Posterior
Probability that the hypothesis is true given the experimental data

Evidence
Normalization factor

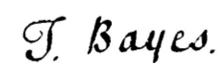
In our case is very simple...

Bayes theorem

Likelihood
Probability that our data describes the hypothesis

Maximum ignorance Prior

$$P(H\,|\,D) \propto \frac{P(D\,|\,H)\cdot P(H)}{P(D)}$$



Posterior
Probability that the hypothesis is true given the experimental data

We only care about proportionality

Thus, assuming a maximum ignorance prior:

$$prob(H \mid D) \propto prob(D \mid H) = L$$

But what is exactly H, in practice?

It is a function and the parameters inside it

$$H_i(a,b) = a + bx_i$$

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Parameter estimation

We would like to estimate the best estimate of a quantity, given some data.
Let's define the posterior as:

$$prob\left(X \mid \{data\}\right) = P$$

The best estimate of X is given by
a maximum in its probability and thus

$$\left.\frac{dP}{dX}\right|_{X_0} = 0$$

We want now to expand P as a Taylor series,
since P varies too fast we take the logarithm of P

$$L = \ln(P) = \ln\left(prob\left(X \mid \{data\}\right)\right)$$

The Taylor expansion of L is:

$$L = L(X_0) + \frac{1}{2}\left.\frac{d^2L}{dX^2}\right|_{X_0}\left(X - X_0\right)^2 + \ldots$$

Taking the first term we get for P:

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[\frac{1}{2}\left.\frac{d^2L}{dX^2}\right|_{X_0}\left(X - X_0\right)^2\right]$$

Let's compare with a gaussian

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[\frac{\left(X - X_0\right)^2}{2\sigma^2}\right]$$

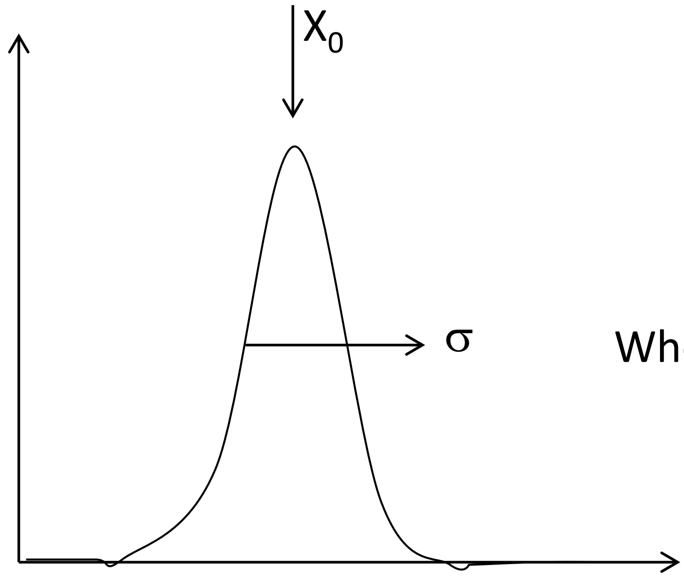
Comparing

Taking the first term we get for P:

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[\frac{1}{2}\frac{d^2L}{dX^2}\bigg|_{X_0}\left(X - X_0\right)^2\right]$$

Let's compare with a gaussian

$$prob\left(X \mid \{data\}\right) \approx A\exp\left[-\frac{\left(X - X_0\right)^2}{2\sigma^2}\right]$$



$$\sigma = \left(-\frac{d^2L}{dX^2}\bigg|_{X_0}\right)^{-1/2}$$

Where $\frac{d^2L}{dX^2}\Big|_{X_0}$ is always negative, so: no problem ;-)

$$X = X_0 \pm \sigma$$

Value plus error comes from a Taylor series

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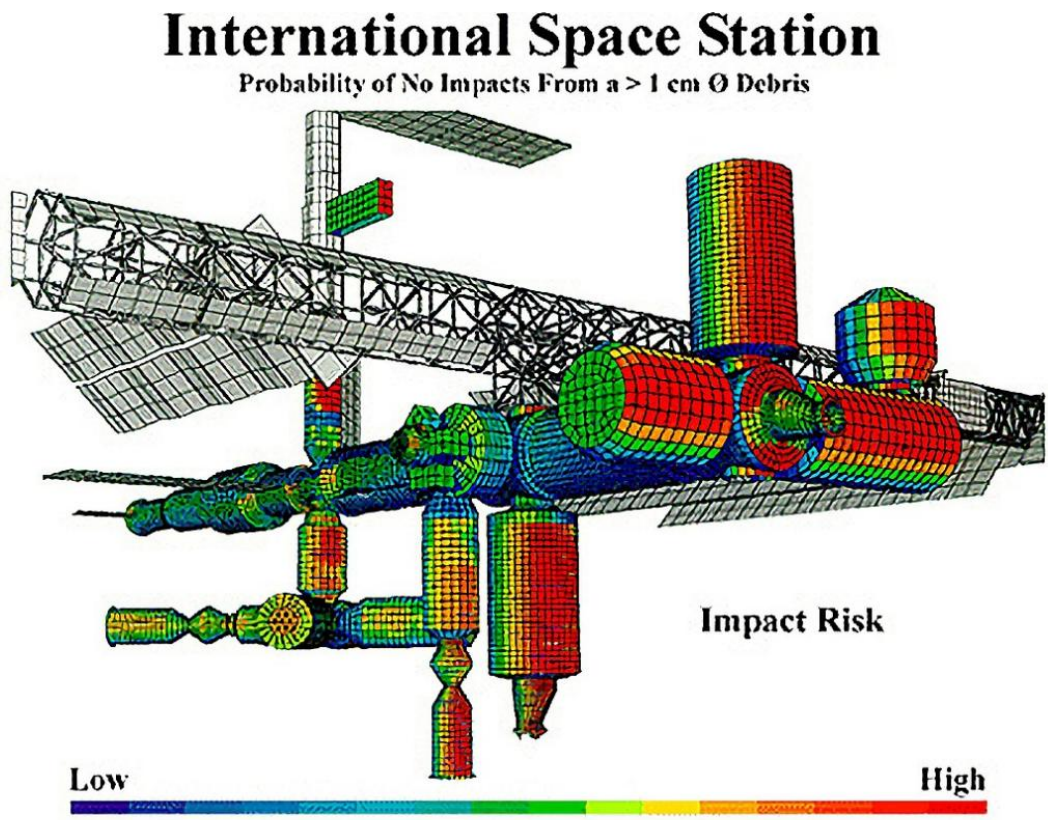
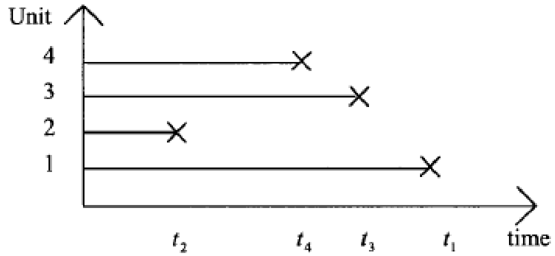
Let's perform tests to quantify risk analysis... before sending the ISS to the space!!!!



Types of tests:

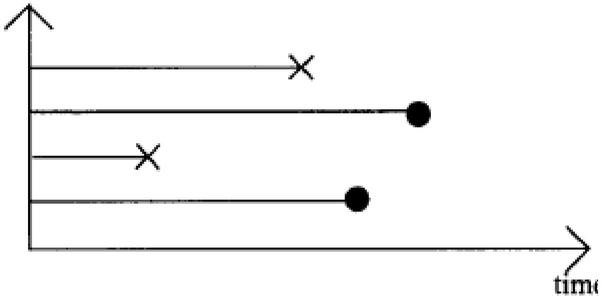
Let’s assume that we perform a test on n=4 components

Complete data sets

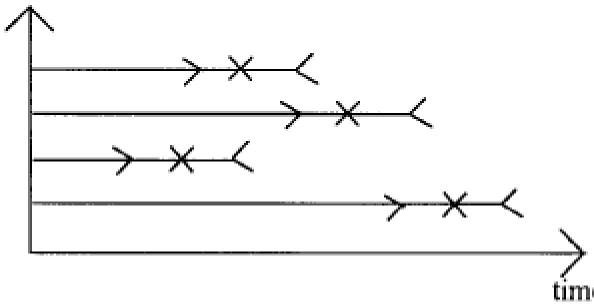


Censored data sets

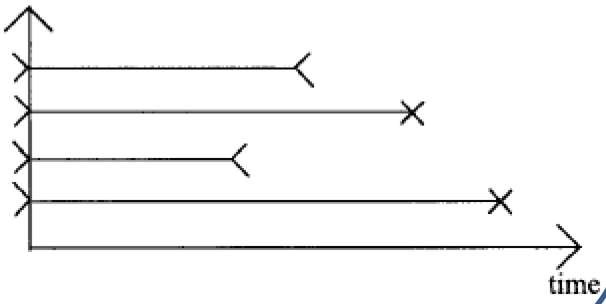
Right censored



Interval censored



Leftcensored

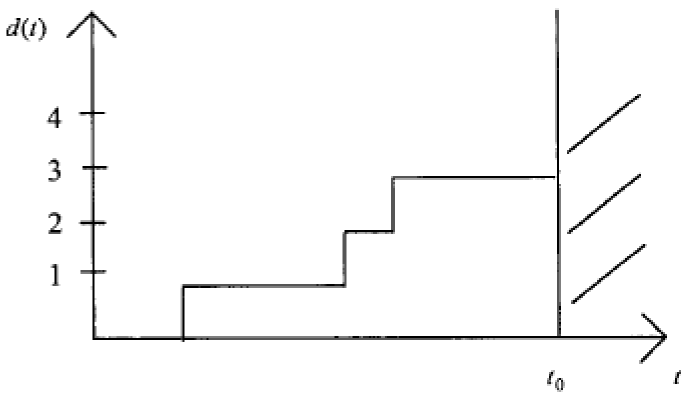


Types of tests:

Test plans

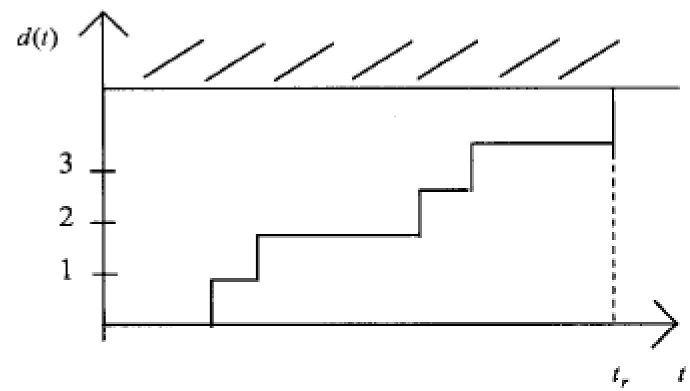
Type I

Testing up to a given time t0

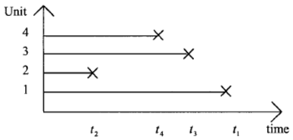


Type II

Testing up to a given r-th failure



They can also be with (R) or witgout (W) replacement



Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***non-censored*** data

We calculate the **likelihood** that

$$L = \prod_{i=1}^{n} \lambda e^{-\lambda t_i} = \lambda e^{-\lambda t_1} \cdot \lambda e^{-\lambda t_2} \cdot \lambda e^{-\lambda t_3} \ldots \lambda e^{-\lambda t_n} = \lambda^n e^{-\lambda T} \leftarrow \boxed{T = \sum_{i=1}^{n} t_i}$$

we take the **logarithm**

$$\ln L = \ln\left(\lambda^n e^{-\lambda T}\right) = n \ln \lambda - \lambda T$$

we make the **derivative**

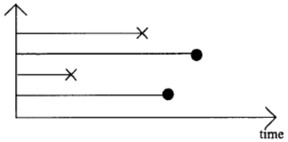
$$\frac{\partial \ln L}{\partial \lambda} = \ln\left(\lambda^n e^{-\lambda T}\right) = \frac{\partial\left(n \ln \lambda - \lambda T\right)}{\partial \lambda} = \frac{n}{\lambda} - T$$

we **set it to zero** (find the maximum)

$$\frac{n}{\hat{\lambda}} - T = 0$$

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{n}{T}$$



Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type I (until t0)

We calculate the **likelihood** that

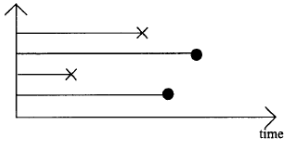
$$L=\underbrace{\prod_{i=1}^{r} f\left(t_i \mid \lambda\right)}_{\text{failure}}\underbrace{\prod_{i=r+1}^{n} R\left(t_0 \mid \lambda\right)}_{\text{time right-censored}}=\lambda^r e^{-\lambda\sum_{i=1}^{r} t_i} e^{-\lambda(n-r)t_0}=\lambda^r e^{-\lambda T} \longleftarrow \boxed{T=\sum_{i=1}^{n} t_i+(n-r)t_0}$$

we take the **logarithm** $\ln L=\ln\left(\lambda^r e^{-\lambda T}\right)=r\ln\lambda-\lambda T$

we make the **derivative** $\dfrac{\partial \ln L}{\partial\lambda}=\ln\left(\lambda^r e^{-\lambda T}\right)=\dfrac{\partial\left(r\ln\lambda-\lambda T\right)}{\partial\lambda}=\dfrac{r}{\lambda}-T$

we **set it to zero** (find the maximum) $\dfrac{r}{\hat{\lambda}}-T=0$

we finally find the **most probable value** for λ $\hat{\lambda}=\dfrac{r}{T}$



Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type II (until r units fail)

We calculate the **likelihood** that

$$L=\prod_{i=1}^{r} \underbrace{f\left(t_i \mid \lambda\right)}_{\text{failure}} \prod_{i=r+1}^{n} \underbrace{R\left(t_i \mid \lambda\right)}_{\text{right-censored}} = \lambda^r e^{-\lambda \sum_{i=1}^{r} t_i} e^{-\lambda \sum_{i=r+1}^{n} t_i} = \lambda^r e^{-\lambda T}$$

$$T=\sum_{i=1}^{n} t_i + (n-r)t_r$$

we take the **logarithm**

$$\ln L = \ln\left(\lambda^r e^{-\lambda T}\right) = r \ln \lambda - \lambda T$$

we make the **derivative**

$$\frac{\partial \ln L}{\partial \lambda} = \ln\left(\lambda^r e^{-\lambda T}\right) = \frac{\partial\left(r \ln \lambda - \lambda T\right)}{\partial \lambda} = \frac{r}{\lambda} - T$$

we **set it to zero** (find the maximum)

$$\frac{r}{\hat{\lambda}} - T = 0$$

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{r}{T}$$

...BUT YOU DO NOT WANT TO STOP PRODUCTION until r components fail

Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type I (until t0)

WITH REPLACEMENT

n=1

Let's do it for a single component **n=1** that is replaced each time it fails

$$L = \lambda e^{-\lambda t_1} \lambda e^{-\lambda(t_2-t_1)} \lambda e^{-\lambda(t_3-t_2)} \ldots \lambda e^{-\lambda(t_0-t_r)} = \lambda^r e^{-\lambda t_0}$$

n>1

... for **n** components

$$L = \lambda^r e^{-\lambda n t_0} = \lambda^r e^{-\lambda T} \leftarrow \boxed{T = n t_0}$$

doing the same calculations as before...

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{r}{T}$$

...BUT YOU DO NOT WANT TO STOP PRODUCTION until r components fail

Determine the failure ratio λ, or its inverse, the **M**ean **T**ime **T**o **F**ailure MTTF τ for ***n*** equal units in the case of ***right-censored*** data type II (until r-th failure)

WITH REPLACEMENT

Let's do it for a single component **n=1** that is replaced each time it fails

n=1

$$L = \lambda e^{-\lambda t_1} \lambda e^{-\lambda(t_2 - t_1)} \lambda e^{-\lambda(t_3 - t_2)} \ldots \lambda e^{-\lambda(t_0 - t_r)} = \lambda^r e^{-\lambda t_0}$$

... for **n** components

n>1

$$L = \lambda^r e^{-\lambda n t_0} = \lambda^r e^{-\lambda T}$$

$T = n t_r$

doing the same calculations as before...

we finally find the **most probable value** for λ

$$\hat{\lambda} = \frac{r}{T}$$

summarizing: We want to keep the notation $\hat{\lambda} = \frac{r}{T}$

so, we change the meaning of T...

not censored, no replacement $T = \sum_{i=1}^{n} t_i$

	Censored Type I (unit t0)	Censored Type II (unit r-th failure)
Without replacement	$T = \sum_{i=1}^{n} t_i + (n-r)t_0$	$T = \sum_{i=1}^{n} t_i + (n-r)t_r$
With replacement	$T = nt_0$	$T = nt_r$