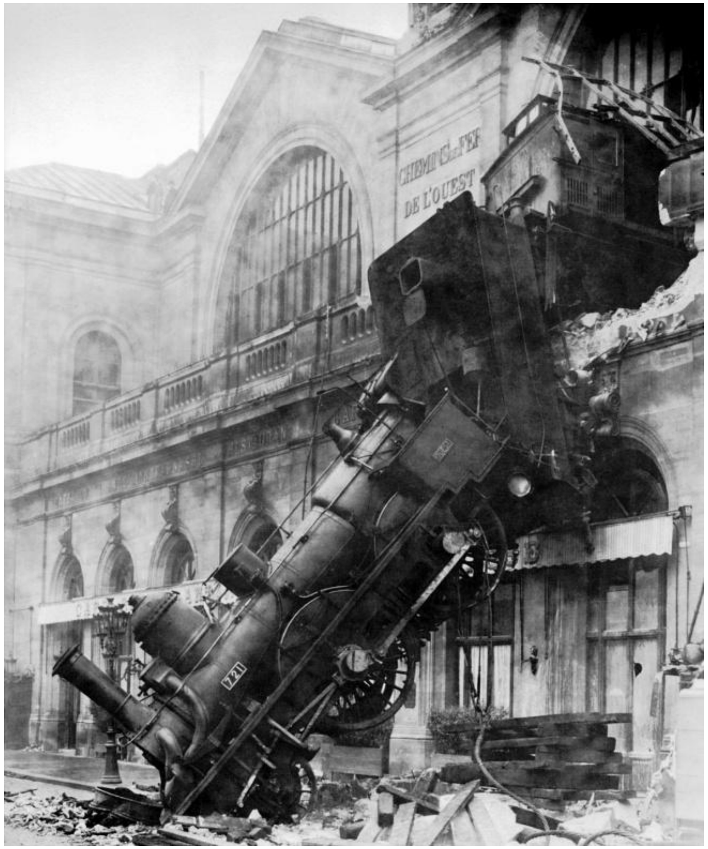
# Unit 3



## Confidence limits

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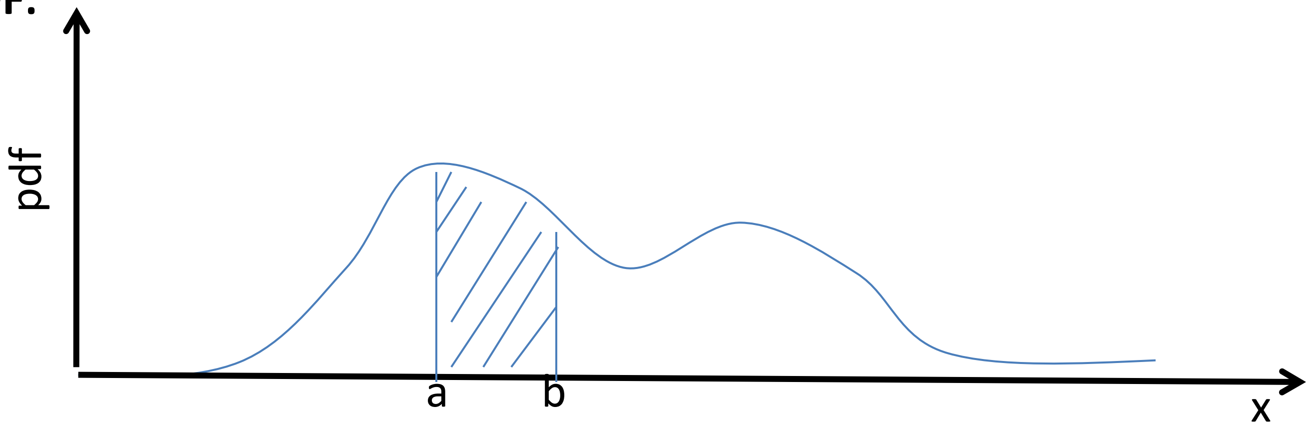
# Summary

1.- Bayes theorem
2.- Maximum likelihood method
3.- Estimation of reliability parameters from tests
4.- Confidence limits of parameters:
normal distribution

# Confidence intervals: the general case

**Let's consider a PDF:**




**Relation with probabilities**

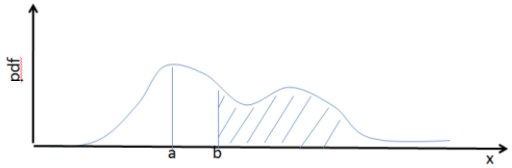
$$p(a < x < b) = \int_a^b f(x)dx$$

**Therefore**

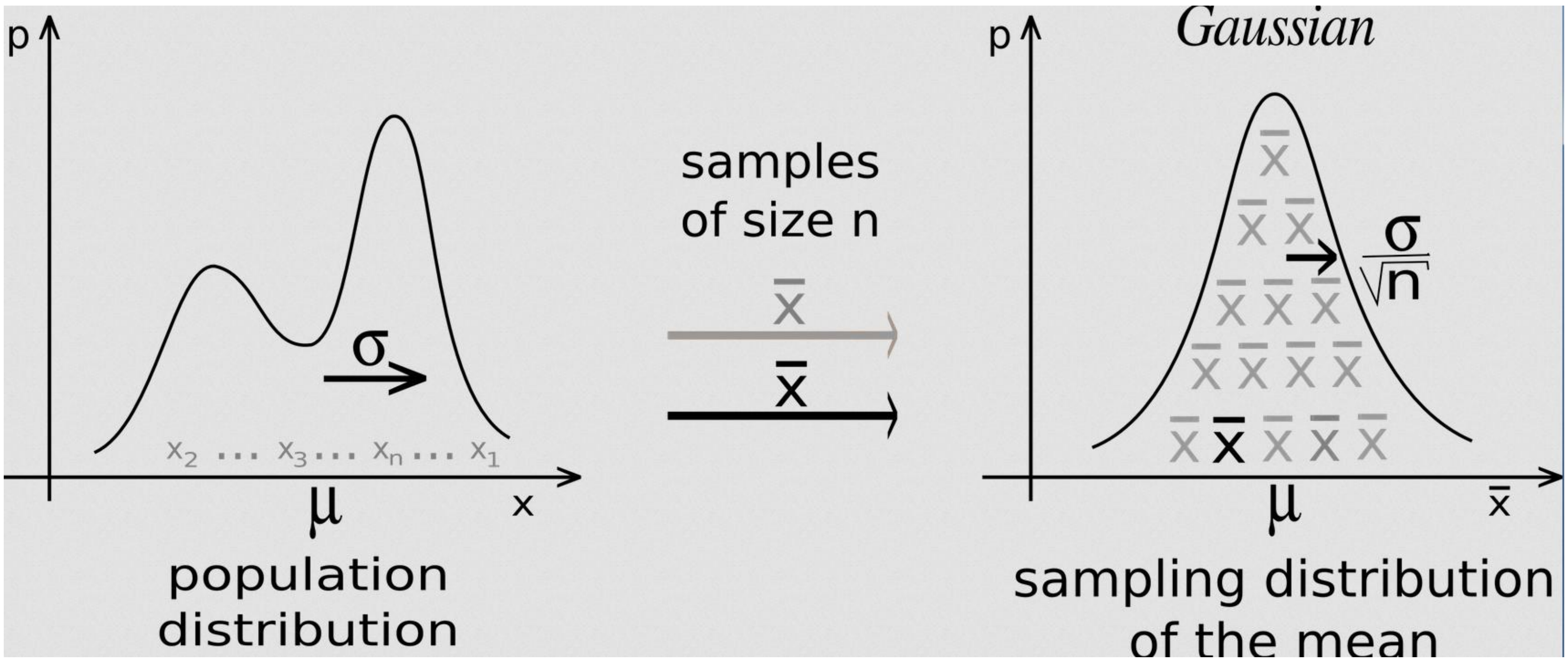
$$\int_{-\infty}^{\infty} f(x) = 1$$

**P-value for b (one side)**

$$p(x > b) = \int_b^{\infty} f(x)dx$$



# Central limit theorem

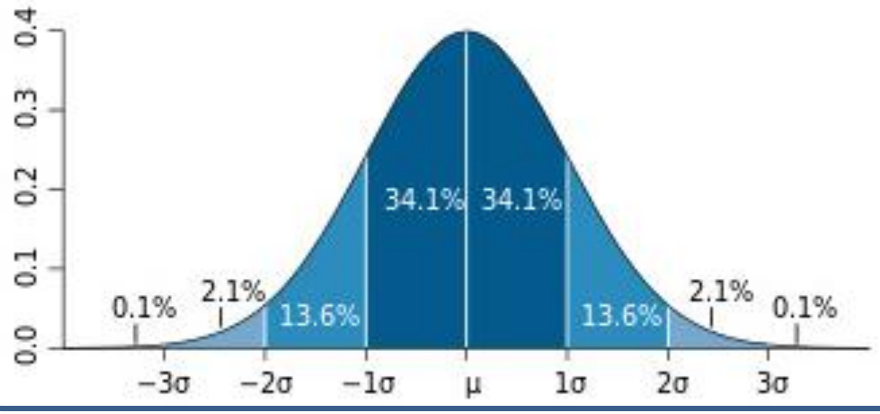



The distribution of the mean tends to be gaussian when increasing n
**no matters the pdf that originated the mean!!!!**

This is the good and the bad thing from statistics is based on the normal PDF!!!!

# Standard Normal distribution

**Standard Normal distribution (μ=0, σ=1)**




$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

Therefore, defining a "z-value" for the whole population as $z = \frac{x-\mu}{\sigma_x}$

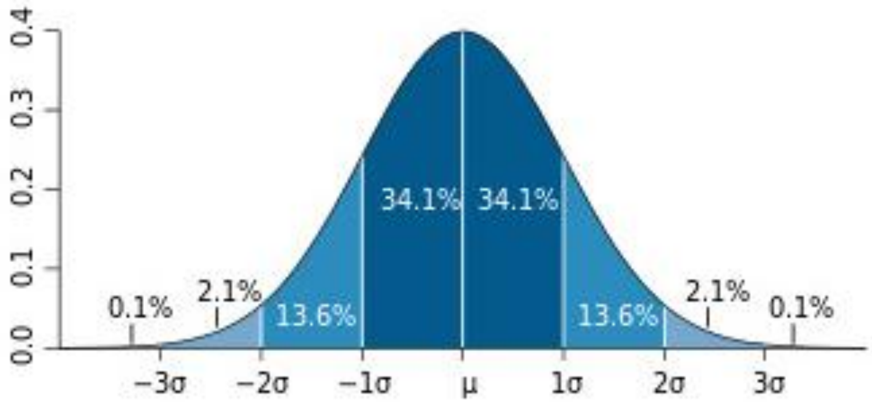
Therefore, defining a "z-value" for n units $z = \frac{\bar{X}-\mu}{\sigma_x / \sqrt{n}}$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2\right) \Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)$$

Z follows a **standard** normal distribution function...

# Confidence intervals: the normal case

**Normal distribution**




$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left(-\frac{(x-\mu)^2}{2\sigma_x^2}\right)$$

$$f(x) \equiv N(\mu, \sigma)$$

**Relation with probabilities**

$$p(-\sigma_x < x < \sigma_x) = \int_{-\sigma}^{\sigma} f(x)dx = 68.2\%$$

**P-value for normal distribution for a value a: is related with the variance**

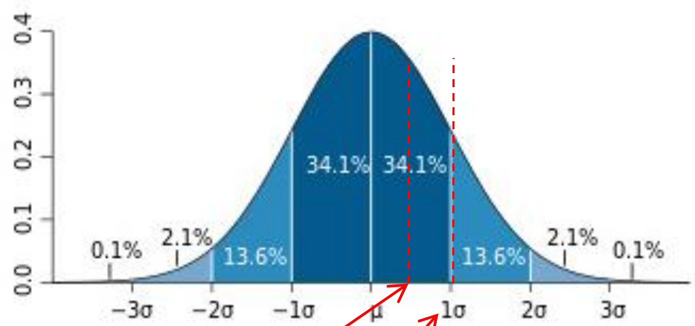
$$p(-\sigma_x < x < \sigma_x) = \int_{-\sigma}^{\sigma} f(x)dx = 68.2\%$$

So we have a relationship between confidence intervals and probability!

There is a 68,2% of chances that your real value is inside your “error “

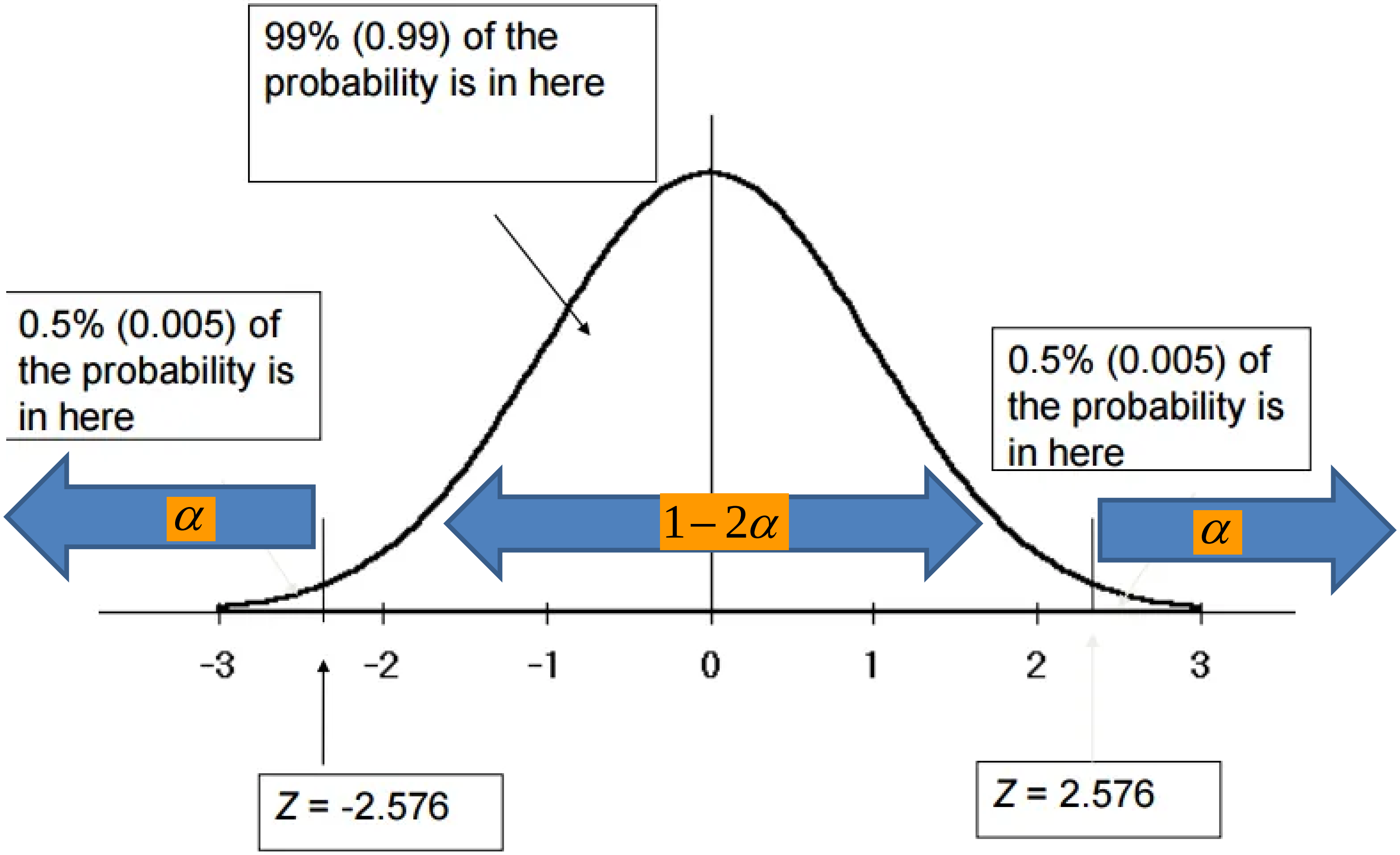
(assuming that x has a normal distribution;-)

standard normal table




| z | +0.00 | +0.01 | +0.02 | +0.03 | +0.04 | +0.05 | +0.06 | +0.07 | +0.08 | +0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.00000 | 0.00399 | 0.00798 | 0.01197 | 0.01595 | 0.01994 | 0.02392 | 0.02790 | 0.03188 | 0.03586 |
| **0.1** | 0.03983 | 0.04380 | 0.04776 | 0.05172 | 0.05567 | 0.05962 | 0.06356 | 0.06749 | 0.07142 | 0.07535 |
| **0.2** | 0.07926 | 0.08317 | 0.08706 | 0.09095 | 0.09483 | 0.09871 | 0.10257 | 0.10642 | 0.11026 | 0.11409 |
| **0.3** | 0.11791 | 0.12172 | 0.12552 | 0.12930 | 0.13307 | 0.13683 | 0.14058 | 0.14431 | 0.14803 | 0.15173 |
| **0.4** | 0.15542 | 0.15910 | 0.16276 | 0.16640 | 0.17003 | 0.17364 | 0.17724 | 0.18082 | 0.18439 | 0.18793 |
| **0.5** | 0.19146 | 0.19497 | 0.19847 | 0.20194 | 0.20540 | 0.20884 | 0.21226 | 0.21566 | 0.21904 | 0.22240 |
| **0.6** | 0.22575 | 0.22907 | 0.23237 | 0.23565 | 0.23891 | 0.24215 | 0.24537 | 0.24857 | 0.25175 | 0.25490 |
| **0.7** | 0.25804 | 0.26115 | 0.26424 | 0.26730 | 0.27035 | 0.27337 | 0.27637 | 0.27935 | 0.28230 | 0.28524 |
| **0.8** | 0.28814 | 0.29103 | 0.29389 | 0.29673 | 0.29955 | 0.30234 | 0.30511 | 0.30785 | 0.31057 | 0.31327 |
| **0.9** | 0.31594 | 0.31859 | 0.32121 | 0.32381 | 0.32639 | 0.32894 | 0.33147 | 0.33398 | 0.33646 | 0.33891 |
| **1.0** | 0.34134 | 0.34375 | 0.34614 | 0.34849 | 0.35083 | 0.35314 | 0.35543 | 0.35769 | 0.35993 | 0.36214 |
| **1.1** | 0.36433 | 0.36650 | 0.36864 | 0.37076 | 0.37286 | 0.37493 | 0.37698 | 0.37900 | 0.38100 | 0.38298 |

# Confidence interval: 1-2α



# Summary

1.- Bayes theorem

2.- Maximum likelihood method

3.- Estimation of reliability parameters from tests

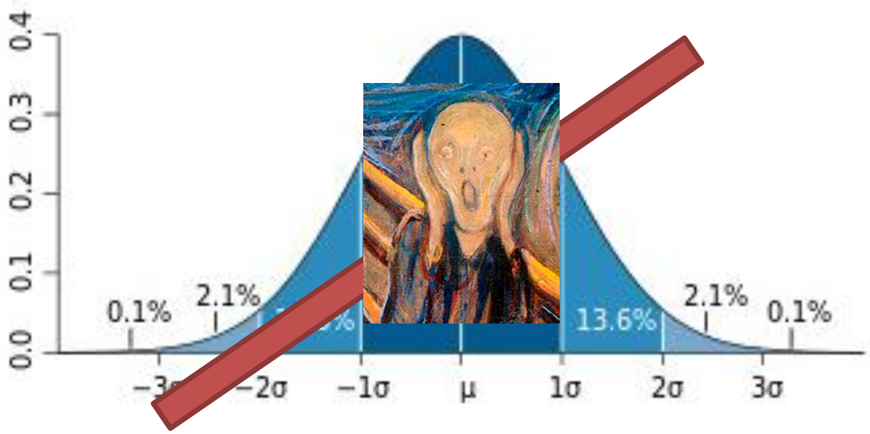
4.- Confidence limits of parameters:

Chi-square distribution

# Problem

failure ratio does not follows a normal distribution!!!

$$\hat{\lambda} = \frac{r}{T}$$




# Solution

... but 2λT does follow a given distribution. Let me proudly introduce you the chi-square distribution

# Chi-Squared distribution

Do you remember the normal distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_x}} \exp\left( -\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2 \right) \Rightarrow f(z) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{2} z^2 \right)$$

We minimize $Z^2$ in order to find $\mu$... the following question arises:

## What is the PDF for $Z^2$ itself?

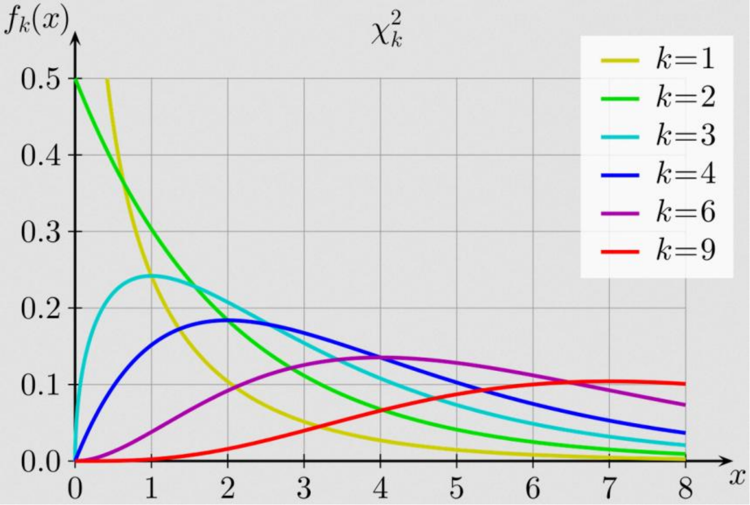
# Chi-Squared distribution

If $X_i$ follow a standard normal distribution then $\chi^2 = \underbrace{X_1^2 + X_2^2 + X_3^2 + \ldots + X_k^2}$

follow a so called $\chi^2$ distribution with **k degrees of freedom** defined as:

$$f(x) = A \cdot x^{k/2-1} e^{-x/2}$$

$$A = \frac{1}{2^{k/2}\Gamma\left(k/2\right)}$$




$$\mu = k; \mathrm{var} = 2k; \max = k - 2$$

If k➔∞ then $\frac{\chi^2 - k}{\sqrt{2k}}$ follows a standard normal distribution , i.e. $\chi^2$ follows N(k,√2k) λ is the failure ratio, and $\theta = 1/\lambda$ is the Mean Time To Failure

T is the TOTAL test time

$2\lambda T$ is $\chi^2$ distributed with **k** degrees of freedom

Type I (fixed $t_0$)
k=2r+2

Type II (fixed r)
k=2r

$2\lambda T$ is $\chi^2$ distributed with **k** degrees of freedom

Type I (fixed $t_0$): k=2r+2

Type II (fixed r): k=2r

Let's now calculate the confidence limits:

ONE SIDED (lower limit)

**Type I (fixed $t_0$)**

k=2r+2

$$P\left[2\lambda T \le \chi^2_\alpha\left(2+2r\right)\right] = \alpha$$

$$P\left[\lambda \le \frac{\chi^2_\alpha\left(2+2r\right)}{2T}\right] = \alpha$$

$$\theta = \frac{1}{\lambda}$$

$$P\left[\theta \le \frac{2T}{\chi^2_\alpha\left(2+2r\right)}\right] = \alpha$$

**Type I (fixed r**

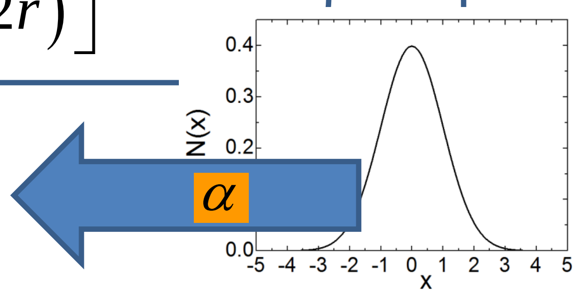
k=2r

$$P\left[2\lambda T \le \chi^2_\alpha\left(2r\right)\right] = \alpha$$

$$P\left[\lambda \le \frac{\chi^2_\alpha\left(2r\right)}{2T}\right] = \alpha$$

$$\theta = \frac{1}{\lambda}$$

$$P\left[\theta \le \frac{2T}{\chi^2_\alpha\left(2r\right)}\right] = \alpha$$



$2\lambda T$ is $\chi^2$ distributed with **k** degrees of freedom

| Type I (fixed $t_0$) | Type II (fixed r) |
| --- | --- |
| k=2r+2 | k=2r |

Let's now calculate the confidence limits:

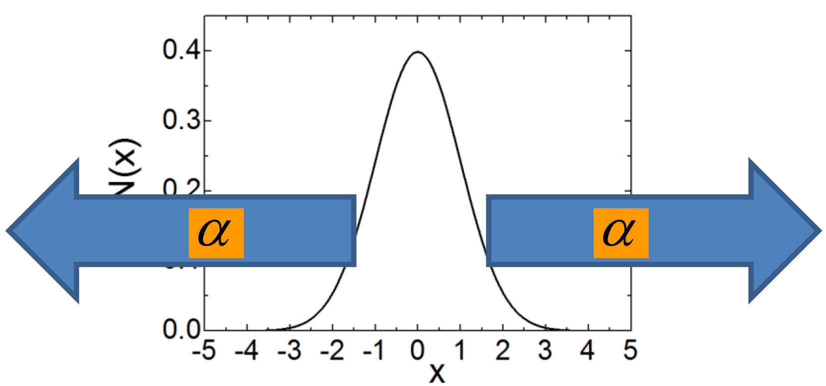
TWO SIDED (lower and upper limit)

Type I (fixed $t_0$)
k=2r+2

$$P\left[\frac{2T}{\chi^2_{\frac{\alpha}{2}}(2+2r)}\theta \le \frac{2T}{\chi^2_{1-\frac{\alpha}{2}}(2+2r)}\right]=\alpha$$

Type I (fixed r
k=2r

$$P\left[\frac{2T}{\chi^2_{\frac{\alpha}{2}}(2r)}\theta \le \frac{2T}{\chi^2_{1-\frac{\alpha}{2}}(2r)}\right]=\alpha$$




4.2, 4.3

## Appendix A: Table of Standard Normal Cumulative Distribution

$$F(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\xi} e^{-\frac{1}{2}x^2}\,dx$$

| $\xi$ | $F(\xi)$ |
|---|---|
| 0.00 | 0.500000 |
| 0.01 | 0.503989 |
| 0.02 | 0.507978 |
| 0.03 | 0.511966 |
| 0.04 | 0.515954 |
| 0.05 | 0.519939 |
| 0.06 | 0.523922 |
| 0.07 | 0.527904 |
| 0.08 | 0.531882 |
| 0.09 | 0.535857 |
| 0.10 | 0.539828 |
| 0.11 | 0.543796 |
| 0.12 | 0.547759 |
| 0.13 | 0.551717 |
| 0.14 | 0.555671 |
| 0.15 | 0.559618 |
| 0.16 | 0.563500 |
| 0.17 | 0.567494 |
| 0.18 | 0.571423 |
| 0.19 | 0.575345 |
| 0.20 | 0.579260 |
| 0.21 | 0.583166 |
| 0.22 | 0.587064 |
| 0.23 | 0.590954 |
| 0.24 | 0.549835 |
| 0.25 | 0.598706 |
| 0.26 | 0.602568 |
| 0.27 | 0.606420 |
| 0.28 | 0.610262 |
| 0.29 | 0.614092 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 0.50 | 0.691463 |
| 0.51 | 0.694975 |
| 0.52 | 0.698468 |
| 0.53 | 0.701944 |
| 0.54 | 0.705401 |
| 0.55 | 0.708840 |
| 0.56 | 0.712260 |
| 0.57 | 0.715661 |
| 0.58 | 0.719043 |
| 0.59 | 0.722405 |
| 0.60 | 0.725747 |
| 0.61 | 0.729069 |
| 0.62 | 0.732371 |
| 0.63 | 0.735653 |
| 0.64 | 0.738914 |
| 0.65 | 0.742154 |
| 0.66 | 0.745374 |
| 0.67 | 0.748572 |
| 0.68 | 0.751748 |
| 0.69 | 0.754903 |
| 0.70 | 0.758036 |
| 0.71 | 0.761148 |
| 0.72 | 0.764238 |
| 0.73 | 0.767305 |
| 0.74 | 0.770350 |
| 0.75 | 0.773373 |
| 0.76 | 0.776373 |
| 0.77 | 0.779350 |
| 0.78 | 0.782305 |
| 0.79 | 0.785236 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 1.00 | 0.841345 |
| 1.01 | 0.843752 |
| 1.02 | 0.846136 |
| 1.03 | 0.848495 |
| 1.04 | 0.850830 |
| 1.05 | 0.853141 |
| 1.06 | 0.855428 |
| 1.07 | 0.857690 |
| 1.08 | 0.859929 |
| 1.09 | 0.862143 |
| 1.10 | 0.864334 |
| 1.11 | 0.866500 |
| 1.12 | 0.868643 |
| 1.13 | 0.870762 |
| 1.14 | 0.872857 |
| 1.15 | 0.874928 |
| 1.16 | 0.876976 |
| 1.17 | 0.878999 |
| 1.18 | 0.881000 |
| 1.19 | 0.882977 |
| 1.20 | 0.884930 |
| 1.21 | 0.886860 |
| 1.22 | 0.888767 |
| 1.23 | 0.890651 |
| 1.24 | 0.892512 |
| 1.25 | 0.894350 |
| 1.26 | 0.896165 |
| 1.27 | 0.897958 |
| 1.28 | 0.899727 |
| 1.29 | 0.901475 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 0.30 | 0.617912 |
| 0.31 | 0.621720 |
| 0.32 | 0.623517 |
| 0.33 | 0.629301 |
| 0.34 | 0.633072 |
| 0.35 | 0.636831 |
| 0.36 | 0.640576 |
| 0.37 | 0.644309 |
| 0.38 | 0.648027 |
| 0.39 | 0.651732 |
| 0.40 | 0.655422 |
| 0.41 | 0.659097 |
| 0.42 | 0.662757 |
| 0.43 | 0.666402 |
| 0.44 | 0.670032 |
| 0.45 | 0.673645 |
| 0.46 | 0.677242 |
| 0.47 | 0.680823 |
| 0.48 | 0.684387 |
| 0.49 | 0.687933 |
| 1.50 | 0.933193 |
| 1.51 | 0.934478 |
| 1.52 | 0.935744 |
| 1.53 | 0.936992 |
| 1.54 | 0.938220 |
| 1.55 | 0.939429 |
| 1.56 | 0.940620 |
| 1.57 | 0.941792 |
| 1.58 | 0.942947 |
| 1.59 | 0.944083 |
| 1.60 | 0.945201 |
| 1.61 | 0.946301 |
| 1.62 | 0.947384 |
| 1.63 | 0.948449 |
| 1.64 | 0.949497 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 0.80 | 0.788145 |
| 0.81 | 0.791030 |
| 0.82 | 0.793892 |
| 0.83 | 0.796731 |
| 0.84 | 0.799546 |
| 0.85 | 0.802337 |
| 0.86 | 0.805105 |
| 0.87 | 0.807850 |
| 0.88 | 0.810570 |
| 0.89 | 0.813267 |
| 0.90 | 0.815940 |
| 0.91 | 0.818589 |
| 0.92 | 0.821214 |
| 0.93 | 0.823815 |
| 0.94 | 0.826391 |
| 0.95 | 0.828944 |
| 0.96 | 0.831473 |
| 0.97 | 0.833977 |
| 0.98 | 0.836457 |
| 0.99 | 0.838913 |
| 2.00 | 0.977250 |
| 2.01 | 0.977784 |
| 2.02 | 0.978308 |
| 2.03 | 0.978822 |
| 2.04 | 0.979325 |
| 2.05 | 0.979818 |
| 2.06 | 0.980301 |
| 2.07 | 0.980774 |
| 2.08 | 0.981237 |
| 2.09 | 0.981691 |
| 2.10 | 0.982136 |
| 2.11 | 0.982571 |
| 2.12 | 0.982997 |
| 2.13 | 0.983414 |
| 2.14 | 0.983823 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 1.30 | 0.903199 |
| 1.31 | 0.904902 |
| 1.32 | 0.906583 |
| 1.33 | 0.908241 |
| 1.34 | 0.909877 |
| 1.35 | 0.911492 |
| 1.36 | 0.913085 |
| 1.37 | 0.914656 |
| 1.38 | 0.916207 |
| 1.39 | 0.917735 |
| 1.40 | 0.919243 |
| 1.41 | 0.920730 |
| 1.42 | 0.922196 |
| 1.43 | 0.923641 |
| 1.44 | 0.925066 |
| 1.45 | 0.926471 |
| 1.46 | 0.927855 |
| 1.47 | 0.929219 |
| 1.48 | 0.930563 |
| 1.49 | 0.931888 |
| 2.50 | 0.993790 |
| 2.51 | 0.993963 |
| 2.52 | 0.994132 |
| 2.53 | 0.994267 |
| 2.54 | 0.994457 |
| 2.55 | 0.994614 |
| 2.56 | 0.994766 |
| 2.57 | 0.994915 |
| 2.58 | 0.995060 |
| 2.59 | 0.995201 |
| 2.60 | 0.995339 |
| 2.61 | 0.995473 |
| 2.62 | 0.995604 |
| 2.63 | 0.995731 |
| 2.64 | 0.995855 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 1.65 | 0.950529 |
| 1.66 | 0.951543 |
| 1.67 | 0.952540 |
| 1.68 | 0.953521 |
| 1.69 | 0.954486 |
| 1.70 | 0.955435 |
| 1.71 | 0.956367 |
| 1.72 | 0.957284 |
| 1.73 | 0.958185 |
| 1.74 | 0.959071 |
| 1.75 | 0.959941 |
| 1.76 | 0.960796 |
| 1.77 | 0.961636 |
| 1.78 | 0.962426 |
| 1.79 | 0.963273 |
| 1.80 | 0.964070 |
| 1.81 | 0.964852 |
| 1.82 | 0.965621 |
| 1.83 | 0.966375 |
| 1.84 | 0.967116 |
| 1.85 | 0.967843 |
| 1.86 | 0.968557 |
| 1.87 | 0.969258 |
| 1.88 | 0.969946 |
| 1.89 | 0.970621 |
| 1.90 | 0.971284 |
| 1.91 | 0.971933 |
| 1.92 | 0.972571 |
| 1.93 | 0.973197 |
| 1.94 | 0.973810 |
| 1.95 | 0.974412 |
| 1.96 | 0.975002 |
| 1.97 | 0.975581 |
| 1.98 | 0.976148 |
| 1.99 | 0.976705 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 2.15 | 0.984223 |
| 2.16 | 0.984614 |
| 2.17 | 0.984997 |
| 2.18 | 0.985371 |
| 2.19 | 0.985738 |
| 2.20 | 0.986097 |
| 2.21 | 0.986447 |
| 2.22 | 0.986791 |
| 2.23 | 0.987126 |
| 2.24 | 0.987455 |
| 2.25 | 0.987776 |
| 2.26 | 0.988089 |
| 2.27 | 0.988396 |
| 2.28 | 0.988696 |
| 2.29 | 0.988989 |
| 2.30 | 0.989276 |
| 2.31 | 0.989556 |
| 2.32 | 0.989830 |
| 2.33 | 0.990097 |
| 2.34 | 0.990358 |
| 2.35 | 0.990613 |
| 2.36 | 0.990863 |
| 2.37 | 0.991106 |
| 2.38 | 0.991344 |
| 2.39 | 0.991576 |
| 2.40 | 0.991802 |
| 2.41 | 0.992024 |
| 2.42 | 0.992240 |
| 2.43 | 0.992451 |
| 2.44 | 0.992656 |
| 2.45 | 0.992857 |
| 2.46 | 0.993053 |
| 2.47 | 0.993244 |
| 2.48 | 0.993431 |
| 2.49 | 0.993613 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 2.65 | 0.995975 |
| 2.66 | 0.996093 |
| 2.67 | 0.996207 |
| 2.68 | 0.996319 |
| 2.69 | 0.996427 |
| 2.70 | 0.996533 |
| 2.71 | 0.996636 |
| 2.72 | 0.996736 |
| 2.73 | 0.996833 |
| 2.74 | 0.996928 |
| 2.75 | 0.997020 |
| 2.76 | 0.997110 |
| 2.77 | 0.997197 |
| 2.78 | 0.997282 |
| 2.79 | 0.997365 |
| 2.80 | 0.997445 |
| 2.81 | 0.997523 |
| 2.82 | 0.997599 |
| 2.83 | 0.997673 |
| 2.84 | 0.997744 |
| 2.85 | 0.997814 |
| 2.86 | 0.997882 |
| 2.87 | 0.997948 |
| 2.88 | 0.998012 |
| 2.89 | 0.998074 |
| 2.90 | 0.998134 |
| 2.91 | 0.998193 |
| 2.92 | 0.998250 |
| 2.93 | 0.998305 |
| 2.94 | 0.998359 |
| 2.95 | 0.998411 |
| 2.96 | 0.998462 |
| 2.97 | 0.998511 |
| 2.98 | 0.998559 |
| 2.99 | 0.998605 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 3.00 | 0.998630 |
| 3.01 | 0.998694 |
| 3.02 | 0.998736 |
| 3.03 | 0.998777 |
| 3.04 | 0.998817 |
| 3.05 | 0.998856 |
| 3.06 | 0.998893 |
| 3.07 | 0.998930 |
| 3.08 | 0.998965 |
| 3.09 | 0.998999 |
| 3.10 | 0.999032 |
| 3.11 | 0.999065 |
| 3.12 | 0.999096 |
| 3.13 | 0.999126 |
| 3.14 | 0.999155 |
| 3.15 | 0.992184 |
| 3.16 | 0.999119 |
| 3.17 | 0.999238 |
| 3.18 | 0.999264 |
| 3.19 | 0.999289 |
| 3.20 | 0.999313 |
| 3.21 | 0.999336 |
| 3.22 | 0.999359 |
| 3.23 | 0.999381 |
| 3.24 | 0.999402 |
| 3.25 | 0.999423 |
| 3.26 | 0.999443 |
| 3.27 | 0.999462 |
| 3.28 | 0.999481 |
| 3.29 | 0.999499 |
| 3.30 | 0.999516 |
| 3.31 | 0.999533 |
| 3.32 | 0.999550 |
| 3.33 | 0.999566 |
| 3.34 | 0.999581 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 3.50 | 0.999767 |
| 3.51 | 0.999776 |
| 3.52 | 0.999784 |
| 3.53 | 0.999792 |
| 3.54 | 0.999800 |
| 3.55 | 0.999807 |
| 3.56 | 0.999815 |
| 3.57 | 0.999821 |
| 3.58 | 0.999828 |
| 3.59 | 0.999835 |
| 3.60 | 0.999841 |
| 3.61 | 0.999847 |
| 3.62 | 0.999853 |
| 3.63 | 0.999858 |
| 3.64 | 0.999864 |
| 3.65 | 0.999869 |
| 3.66 | 0.999874 |
| 3.67 | 0.999879 |
| 3.68 | 0.999883 |
| 3.69 | 0.999888 |
| 3.70 | 0.999892 |
| 3.71 | 0.999806 |
| 3.72 | 0.999900 |
| 3.73 | 0.999904 |
| 3.74 | 0.999908 |
| 3.75 | 0.999912 |
| 3.76 | 0.999915 |
| 3.77 | 0.999918 |
| 3.78 | 0.999922 |
| 3.79 | 0.999925 |
| 3.80 | 0.999928 |
| 3.81 | 0.999931 |
| 3.82 | 0.999933 |
| 3.83 | 0.999936 |
| 3.84 | 0.999938 |

| $\xi$ | $1-F(\xi)$ |
|---|---|
| 4.00 | 0.316712E-04 |
| 4.05 | 0.256088E-04 |
| 4.10 | 0.206575E-04 |
| 4.15 | 0.166238E-04 |
| 4.20 | 0.133458E-04 |
| 4.25 | 0.106883E-04 |
| 4.30 | 0.853906E-05 |
| 4.35 | 0.680688E-05 |
| 4.40 | 0.541234E-05 |
| 4.45 | 0.429351E-05 |
| 4.50 | 0.339767E-05 |
| 4.55 | 0.268230E-05 |
| 4.60 | 0.211245E-05 |
| 4.65 | 0.165968E-05 |
| 4.70 | 0.130081E-05 |
| 4.75 | 0.101708E-05 |
| 4.80 | 0.793328E-06 |
| 4.85 | 0.617307E-06 |
| 4.90 | 0.479183E-06 |
| 4.95 | 0.371067E-06 |
| 5.00 | 0.286652E-06 |
| 5.10 | 0.169827E-06 |
| 5.20 | 0.996443E-07 |
| 5.30 | 0.579013E-07 |
| 5.40 | 0.333204E-07 |
| 5.50 | 0.189896E-07 |
| 5.60 | 0.107176E-07 |
| 5.70 | 0.599037E-08 |
| 5.80 | 0.331575E-08 |
| 5.90 | 0.181751E-08 |
| 6.00 | 0.986588E-09 |
| 6.10 | 0.530343E-09 |
| 6.20 | 0.282316E-09 |
| 6.30 | 0.148823E-09 |
| 6.40 | 0.77688 E-10 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 3.35 | 0.999596 |
| 3.36 | 0.999610 |
| 3.37 | 0.999624 |
| 3.38 | 0.999637 |
| 3.39 | 0.999650 |
| | |
| 3.40 | 0.999663 |
| 3.41 | 0.999675 |
| 3.42 | 0.999687 |
| 3.43 | 0.999698 |
| 3.44 | 0.999709 |
| | |
| 3.45 | 0.999720 |
| 3.46 | 0.999730 |
| 3.47 | 0.999740 |
| 3.48 | 0.999749 |
| 3.49 | 0.999758 |

| $\xi$ | $F(\xi)$ |
|---|---|
| 3.85 | 0.999941 |
| 3.86 | 0.999943 |
| 3.87 | 0.999946 |
| 3.88 | 0.999948 |
| 3.89 | 0.999950 |
| | |
| 3.90 | 0.999952 |
| 3.91 | 0.999954 |
| 3.92 | 0.999956 |
| 3.93 | 0.999958 |
| 3.94 | 0.999959 |
| | |
| 3.95 | 0.999961 |
| 3.96 | 0.999963 |
| 3.97 | 0.999964 |
| 3.98 | 0.999966 |
| 3.99 | 0.999967 |

| $\xi$ | $1-F(\xi)$ |
|---|---|
| 6.50 | 0.40160 E-10 |
| 6.60 | 0.20558 E-10 |
| 6.70 | 0.10421 E-10 |
| 6.80 | 0.5231 E-11 |
| 6.90 | 0.260 E-11 |
| | |
| 7.00 | 0.128 E-11 |
| 7.10 | 0.624 E-12 |
| 7.20 | 0.301 E-12 |
| 7.30 | 0.144 E-12 |
| 7.40 | 0.68 E-13 |
| | |
| 7.50 | 0.32 E-13 |
| 7.60 | 0.15 E-13 |
| 7.70 | 0.70 E-14 |
| 7.80 | 0.30 E-14 |
| 7.90 | 0.15 E-14 |

## Appendix B: Table of Chi-Square Cumulative Distribution

$\chi_{\alpha}^{2}(f)$

| f\ $\alpha$ | 0.995 | 0.99 | 0.975 | 0.95 | 0.90 | 0.10 | 0.05 | 0.025 | 0.01 | 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|
| **1** | —— | —— | 0.001 | 0.004 | 0.016 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 |
| **2** | 0.010 | 0.020 | 0.051 | 0.103 | 0.211 | 4.605 | 5.991 | 7.378 | 9.210 | 10.597 |
| **3** | 0.072 | 0.115 | 0.216 | 0.352 | 0.584 | 6.251 | 7.815 | 9.348 | 11.345 | 12.838 |
| **4** | 0.207 | 0.297 | 0.484 | 0.711 | 1.064 | 7.779 | 9.488 | 11.143 | 13.277 | 14.860 |
| **5** | 0.412 | 0.554 | 0.831 | 1.145 | 1.610 | 9.236 | 11.070 | 12.833 | 15.086 | 16.750 |
| **6** | 0.676 | 0.872 | 1.237 | 1.635 | 2.204 | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| **7** | 0.989 | 1.239 | 1.690 | 2.167 | 2.833 | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| **8** | 1.344 | 1.646 | 2.180 | 2.733 | 3.490 | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| **9** | 1.735 | 2.088 | 2.700 | 3.325 | 4.168 | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| **10** | 2.156 | 2.558 | 3.247 | 3.940 | 4.865 | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| **11** | 2.603 | 3.053 | 3.816 | 4.575 | 5.578 | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| **12** | 3.074 | 3.571 | 4.404 | 5.226 | 6.304 | 18.549 | 21.026 | 23.337 | 26.217 | 28.300 |
| **13** | 3.565 | 4.107 | 5.009 | 5.892 | 7.042 | 19.812 | 22.362 | 24.736 | 27.688 | 29.819 |
| **14** | 4.075 | 4.660 | 5.629 | 6.571 | 7.790 | 21.064 | 23.685 | 26.119 | 29.141 | 31.319 |
| **15** | 4.601 | 5.229 | 6.262 | 7.261 | 8.547 | 22.307 | 24.996 | 27.488 | 30.578 | 32.801 |
| **16** | 5.142 | 5.812 | 6.908 | 7.962 | 9.312 | 23.542 | 26.296 | 28.845 | 32.000 | 34.267 |
| **17** | 5.697 | 6.408 | 7.564 | 8.672 | 10.085 | 24.769 | 27.587 | 30.191 | 33.409 | 35.718 |
| **18** | 6.265 | 7.015 | 8.231 | 9.390 | 10.865 | 25.989 | 28.869 | 31.526 | 34.805 | 37.156 |
| **19** | 6.844 | 7.633 | 8.907 | 10.117 | 11.651 | 27.204 | 30.144 | 32.852 | 36.191 | 38.582 |
| **20** | 7.434 | 8.260 | 9.591 | 10.851 | 12.443 | 28.412 | 31.410 | 34.170 | 37.566 | 39.997 |
| **21** | 8.034 | 8.897 | 10.283 | 11.591 | 13.240 | 29.615 | 32.671 | 35.479 | 38.932 | 41.401 |
| **22** | 8.643 | 9.542 | 10.982 | 12.338 | 14.041 | 30.813 | 33.924 | 36.781 | 40.289 | 42.796 |
| **23** | 9.260 | 10.196 | 11.689 | 13.091 | 14.848 | 32.007 | 35.172 | 38.076 | 41.638 | 44.181 |
| **24** | 9.886 | 10.856 | 12.401 | 13.848 | 15.659 | 33.196 | 36.415 | 39.364 | 42.980 | 45.559 |
| **25** | 10.520 | 11.524 | 13.120 | 14.611 | 16.473 | 34.382 | 37.652 | 40.646 | 44.314 | 46.928 |
| **26** | 11.160 | 12.198 | 13.844 | 15.379 | 17.292 | 35.563 | 38.885 | 41.923 | 45.642 | 48.290 |
| **27** | 11.808 | 12.879 | 14.573 | 16.151 | 18.114 | 36.741 | 40.113 | 43.195 | 46.963 | 49.645 |
| **28** | 12.461 | 13.565 | 15.308 | 16.928 | 18.939 | 37.916 | 41.337 | 44.461 | 48.278 | 50.993 |
| **29** | 13.121 | 14.256 | 16.047 | 17.708 | 19.768 | 39.087 | 42.557 | 45.722 | 49.588 | 52.336 |
| **30** | 13.787 | 14.953 | 16.791 | 18.493 | 20.599 | 40.256 | 43.773 | 46.979 | 50.892 | 53.672 |
| **40** | 20.707 | 22.164 | 24.433 | 26.509 | 29.051 | 51.805 | 55.758 | 59.342 | 63.691 | 66.766 |
| **50** | 27.991 | 29.707 | 32.357 | 34.764 | 37.689 | 63.167 | 67.505 | 71.420 | 76.154 | 79.490 |
| **60** | 35.534 | 37.485 | 40.482 | 43.188 | 46.459 | 74.397 | 79.082 | 83.298 | 88.379 | 91.952 |
| **70** | 43.275 | 45.442 | 48.758 | 51.739 | 55.329 | 85.527 | 90.531 | 95.023 | 100.425 | 104.215 |
| **80** | 51.172 | 53.540 | 57.153 | 60.391 | 64.278 | 96.578 | 101.879 | 106.629 | 112.329 | 116.321 |
| **90** | 59.196 | 61.754 | 65.647 | 69.126 | 73.291 | 107.565 | 113.145 | 118.136 | 124.116 | 128.299 |
| **100** | 67.328 | 70.065 | 74.222 | 77.929 | 82.358 | 118.498 | 124.342 | 129.561 | 135.807 | 140.169 |

$$f(x) = A \cdot x^{k/2-1} e^{-x/2}$$

$$A = \frac{1}{2^{k/2}\,\Gamma\left(k/2\right)}$$