Microscopic constituents of matter

All matter except for that created in big accelerators is made of <u>electrons</u>, <u>protons</u> & <u>neutrons</u>
Of these 3 particles, two have charge, and all have "spin" (intrinsic magnetic moment that produce); moreover, as they move they generate an extra magnetic field:

all matter is made of tiny charges and magnets

In fact charge and spin are in some sense more fundamental than mass: mass is not quantized, varies depending on the reference frame (special relativity), and is the effect of an interaction with a field (Higgs particle); on the contrary, charge and spin are quantized and invariant.
The spin is described classically as a magnetic dipole moment (a little magnet with a south pole and a north pole right next to one another, topic 3). This is the magnetic version of an electric dipole moment (topics 1&2), which is a configuration of opposite plus and minus charges right next to one another. This configuration of charges (electric dipole) also exist in nature, not in particles by themselves but in their agglomerates: the water molecule is an electric dipole.
The microscopic constituents of matter give rise to E-fields and B-fields (and are subject to external E- and B- fields). The simplest description could be with the following equations:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2} \quad \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\vec{p}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{p}}{r^3}\right) \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}q\frac{\vec{v}\times\hat{r}}{r^2} \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{m}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{m}}{r^3}\right)$$

E-field of (point)charge & electric dipole   *B-field of <u>slowly moving</u> charge & magnetic dipole*

plus the Lorentz force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ and the torques on the dipoles $\begin{cases} \vec{\Gamma} = \vec{p}\times\vec{E} \\ \vec{\Gamma} = \vec{m}\times\vec{B} \end{cases}$

NOTE: Charges are depicted as points, dipoles (both electric and magnetic) are depicted as arrows. So in the first three topics it will be all about points, arrows, spheres, cylinders… ☺

Why do we settle for "less"?

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau'\,\rho(\vec{r}')\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \qquad vs \qquad \begin{cases} \oint_{\partial Vol}\vec{E}\cdot d\vec{a} = Q_{inside}/\varepsilon_0 \\ \oint_{\partial Area}\vec{E}\cdot d\vec{\ell} = 0 \end{cases}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau'\,\vec{J}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \qquad vs \qquad \begin{cases} \oint_{\partial Vol}\vec{B}\cdot d\vec{a} = 0 \\ \oint_{\partial Area}\vec{B}\cdot d\vec{\ell} = \mu_0 I \end{cases}$$

1)   Because <u>these expressions</u> cannot be generalized to **time-varying fields**, while <u>these</u> can;
In fact, the equations for the fluxes (Gauss law) are correct even for time-dependent fields
2) Because field equations (with fluxes and line integrals) have a profound meaning (see next slide)
3) They simplify the solution of certain problems:
    3.1) high-symmetry charge distributions
    3.2) fields near interfaces ("boundary conditions") →
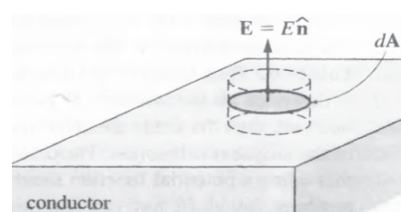
$$E_n = \frac{\sigma}{\varepsilon_0}$$

4) Helmholtz theorem states that **a field that only depends on *r* (and not time) is completely specified if both its curl and divergence are known**. Hence we are not really settling for less!
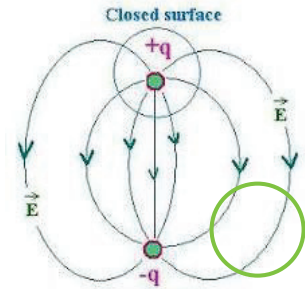
Physical meaning of Gauss's and Ampère's law for E & B (electro- & magneto-statics, física 2)

$$\oint_{\partial Vol} \vec{E} \cdot d\vec{a} = \frac{Q_{inside}}{\varepsilon_0}$$

$$(\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0)$$

means that **E-field lines start at positive charges and end at negative charges**
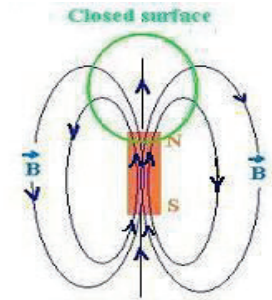
Example: electric dipole →

$$\oint_{\partial Vol} \vec{B} \cdot d\vec{a} = 0$$

$$(\vec{\nabla} \cdot \vec{B} = 0)$$

means that **there exist no magnetic monopoles** (no positive and negative "charges" or poles exist that are source of B). **B field lines never start nor end**

Example: bar magnet →

$$\oint_{\partial Area} \vec{E} \cdot d\vec{\ell} = 0$$

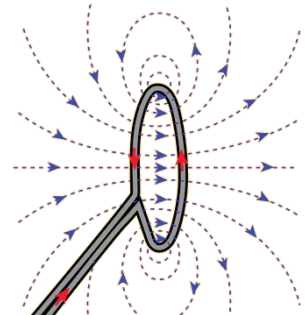$$(\vec{\nabla} \times \vec{E} = 0)$$

means that **E-field lines never draw a closed contour**, they never make a full circle around any point

$$\oint_{\partial Area} \vec{B} \cdot d\vec{\ell} = \mu_0 I$$

$$(\vec{\nabla} \times \vec{B} = \mu_0 \vec{J})$$

means that **B-field lines draw closed contours around currents**, they "curl" around currents

Example: current flowing in a ring →
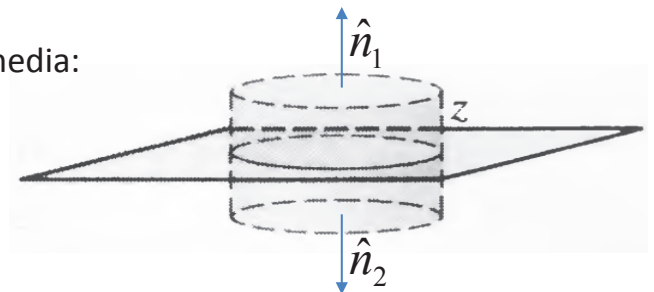
Differential & integral laws & boundary conditions

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \int_{Vol} d\tau \vec{\nabla} \cdot \vec{E} = \int_{Vol} \frac{d\tau \rho}{\varepsilon_0} = \frac{Q_{inside\_Vol}}{\varepsilon_0}$$

Gauss's integral theorem $\int_{Vol} d\tau \vec{\nabla} \cdot \vec{C} = \oint_{\partial Vol} \vec{C} \cdot d\vec{a}$, therefore $\oint_{\partial Vol} \vec{E} \cdot d\vec{a} = \frac{Q_{inside}}{\varepsilon_0}$

If charge is present at the boundary between 2 media:

$$\Phi(\vec{E}) = \vec{E}_1 \cdot \hat{n}_1 A + \vec{E}_2 \cdot \hat{n}_2 A = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

whence $\vec{E}_1 \cdot \hat{n}_1 + \vec{E}_2 \cdot \hat{n}_2 = \frac{\sigma}{\varepsilon_0}$

Hence:

Differential form	Integral form	Boundary form

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \iff \oint_{\partial Vol} \vec{E} \cdot d\vec{a} = \frac{Q_{inside}}{\varepsilon_0} \iff \vec{E}_1 \cdot \hat{n}_1 + \vec{E}_2 \cdot \hat{n}_2 = \frac{\sigma}{\varepsilon_0}$$
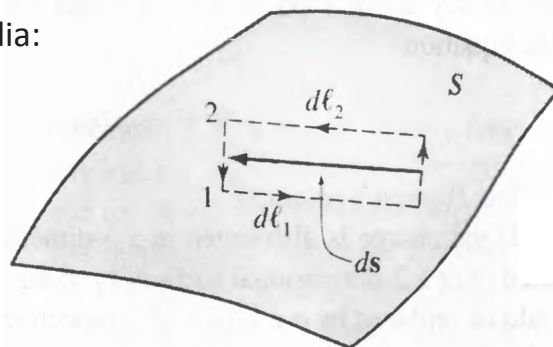
Differential & integral laws & boundary conditions

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad \int_{Area} \left(\vec{\nabla} \times \vec{B}\right) \cdot d\vec{a} = \mu_0 \int_{Area} \vec{J} \cdot d\vec{a} = \mu_0 I_{through\_Area}$$

Stokes' integral theorem $\int_{Area} \left(\vec{\nabla} \times \vec{C}\right) \cdot d\vec{a} = \oint_{\partial Area} \vec{C} \cdot d\vec{\ell}$, therefore $\oint_{\partial Area} \vec{B} \cdot d\vec{\ell} = \mu_0 I$

If a current is present at the boundary between 2 media:

$$\oint \vec{B} \cdot d\vec{\ell} = B_{t2} d\ell_2 + B_{t1} d\ell_1 = (B_{t2} - B_{t1}) |d\ell| =$$
$$= \mu_0 I da = \mu_0 K |d\ell|$$

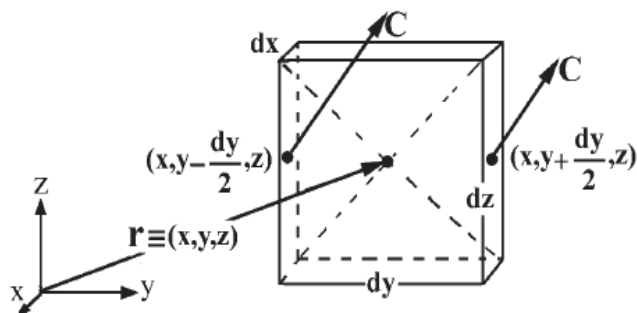whence $\quad \vec{B}_1 \times \hat{n}_1 + \vec{B}_2 \times \hat{n}_2 = \mu_0 \vec{K}$

Hence:

Differential form	Integral form	Boundary form
$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ $\Longleftrightarrow$	$\oint_{\partial Area} \vec{B} \cdot d\vec{\ell} = \mu_0 I$ $\Longleftrightarrow$	$\vec{B}_1 \times \hat{n}_1 + \vec{B}_2 \times \hat{n}_2 = \mu_0 \vec{K}$

Gauss's integral theorem

Take a little volume of sides dx, dy, dz, centered at position **r** = (x, y, z)

We want to calculate the flux of the vector field **C** = (C_x , C_y , C_z) through the volume's boundary

Consider the flux through the two vertical facets (parallel to the xy plane). The flux through the right facet is $C_y(x, y + dy/2, z)\, dxdz$; the flux through the left facet is $-C_y(x, y - dy/2, z)\, dxdz$. The minus sign comes from the fact that the flux of a vector is by definition the outward flux. By making a Taylor expansion to first order, we get that the total flux through both xz facets is:

$$\left[C_y\left(x, y + \frac{dy}{2}, z\right) - C_y\left(x, y - \frac{dy}{2}, z\right)\right] dxdz = \left|C_y(x,y,z) + \frac{\partial C_y(x,y,z)}{\partial y}\frac{dy}{2} - C_y(x,y,z) - \frac{\partial C_y(x,y,z)}{\partial y}\frac{dy}{2}\right| dxdz = \frac{\partial C_y(x,y,z)}{\partial y} dxdydz$$

the total outward flux is then: $\quad d\Phi = \left|\frac{\partial C_x(x,y,z)}{\partial x} + \frac{\partial C_y(x,y,z)}{\partial y} + \frac{\partial C_z(x,y,z)}{\partial z}\right| dxdydz = \vec{\nabla} \cdot \vec{C}\, dV$

looking at the figure to the right, we see that the outward flux from an inner elementary volume cancels out with the outward flux from its neighbors. Whence we get Gauss's theorem:

$$\Phi_{TOTAL} = d\Phi_1 + d\Phi_2 + d\Phi_3 + \cdots = \int d\Phi = \oint \vec{\nabla} \cdot \vec{C}\, dV$$
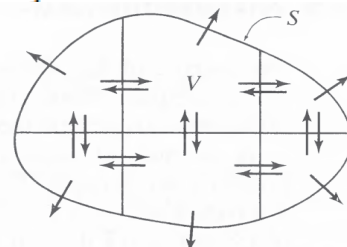
Theoretical tools

The fundamental quantities in electromagnetism are fields (electric field, magnetic field, scalar potential field, etc.), that is, scalar or vector functions of the position **r** (and, when we consider time varying fields, also of time t). To describe these quantities, we will need few concepts and tools from classical field theory, which we will introduce and use during the course:

- Rectangular (cartesian), cylindrical, and spherical spatial coordinates
- Gradient, divergence, curl, time derivative
- Dirac's delta
- Gauss's and Stokes integral theorems
- Boundary conditions
- Derivative rules for products (scalar & vector)
- integration by parts

➢ **MACROSCOPIC DESCRIPTION**
(average)
Connection with: circuit theory, electromagnetic waves, geometric and wave optics (PEF1), statistical physics

→ Maxwell's equations (in terms of ε_r, μ_r, g, n) and boundary conditions

➢ **MICROSCOPIC DESCRIPTION**
Connection with: chemistry, quantum physics, solid state physics, physical electronics, photonics

→ Atomic/solid state description: microscopic constituents, dipoles, inter-particle forces; microscopic origin of values of ε_r, μ_r, g, n

NOTATION

Finite variations: Δ
Derivatives: $d, \partial_t, \partial_x, \vec{\nabla}$

vectors and vector fields:
Position: $\vec{r}$; distance $\vec{d}$
Normal direction $\hat{n}$; velocity $\vec{v}$
Force: $\vec{F}$, torque: $\vec{\Gamma}$
Force per unit charge or volume: $\vec{f}$
Electric dipole moment $\vec{p}$
Magnetic dipole moment $\vec{m}$
Electric field $\vec{E}$
Polarization field $\vec{P}$
(electric line dipole density: $\vec{\wp}$)
Displacement field $\vec{D}$
Magnetic field $\vec{B}$
Magnetization field $\vec{M}$
Auxiliary field $\vec{H}$
Vector potential $\vec{A}$
Current density: (free $_f$, bound $_b$)
- volume $\vec{J}$
- surface $\vec{K}$
- line $\vec{I}$
Poynting vector: $\vec{S}$
Generic vector field: $\vec{G}$
Constant vector (field) : $\vec{C}$

infinitesimals (with d or δ):
- distance: $d\ell$ (dx, dy, ...), $\vec{d\ell} = d\vec{r}$
- surface: da, $\vec{da} = \hat{n}da$
- volume: $d\tau$; solid angle: $d\Omega$
scalars and scalar fields
Number/number density: n, N
Dirac's delta function: δ
Electric potential/voltage: V or ϕ
Field (or potential) energy: U
Total energy: E
(field) energy density: u
Charge: q, Q
Proton/electron charge: $+/- e$
Charge density: (free $_f$, bound $_b$)
- volume ρ
- surface σ
- line λ
Susceptibility: χ
Dielectric permittivity: ε
Magnetic permeability: μ
Magnetic scalar potential: Ξ
Flux: Φ Irradiance: $\Im$
Complex refractive index: $\tilde{n}$

Coordinates:
- rectangular: (x, y, z)
- spherical: (r, θ, φ)
- cylindrical: (z, s, θ)
- unit vectors: $\hat{x}, \hat{y}, \hat{z}, \hat{r}$
   (also $\hat{i}, \hat{s}, \hat{\theta}, ...$)
Mass: m ; Time: t
Temperature: T
Length: L, d
Speed of light: c
Generic constant: k, κ
Surface: S, Area
Volume: Vol
Capacitance: C
Resistance: R
Current: I ; Power: $\wp$
Conductivity: g
Mobility: μ
Resistivity: η
Polarisability: α
Inductance: L
Electromotance: $\mathscr{E}$
Magnetomotance: $\mathscr{M}$
Reluctance: $\mathfrak{R}$
complex permittivity: $\tilde{\varepsilon}$

Summary of electrostatics (topic 1)

Coulomb's law $\vec{F} = \dfrac{1}{4\pi\varepsilon_0} q_1 q_2 \dfrac{\vec{r}}{r^3}$ with $\boxed{\dfrac{1}{4\pi\varepsilon_0} \approx 9\times10^9 \ \text{Nm}^2/\text{C}^2}$

$\vec{E}(r) = \lim\limits_{q\to 0} \dfrac{\vec{F}}{q}$ → Field of a point charge: $\vec{E}(r) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\hat{r}}{r^2} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\vec{r}}{r^3}$

→ Field of a set of point charges: $\vec{E}(\vec{r}) = \sum\limits_{k=1}^{N} \dfrac{q_k}{4\pi\varepsilon_0} \dfrac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}$

→ Field of a continuous distribution of charges:

$dq = \rho\, d\tau$; $\Sigma \to \int$ $\Rightarrow$ $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\int d\tau' \dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}')$

By the identities: $\dfrac{\vec{r}}{r^3} = -\vec{\nabla}\left(\dfrac{1}{r}\right)$ and $\dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}\dfrac{1}{|\vec{r}-\vec{r}'|}$, we see that $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

with $V(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0}\dfrac{1}{r}$ or: $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\sum\limits_{k=1}^{N}\dfrac{q_k}{|\vec{r}-\vec{r}_k|}$ or: $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\int d\tau' \dfrac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$

$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ implies $\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\vec{\nabla}V = 0$

It also implies: $V(\vec{r}) = -\int \vec{E}(\vec{r})\cdot d\vec{\ell}$ and $U_q(\vec{r}) = qV(\vec{r})$

Dirac's delta

Instead of having 2 different types of formulas, for the continuous and discrete cases, we'd like to have only 1

$\vec{E}(\vec{r}) = \sum\limits_{k=1}^{N} \dfrac{q_k}{4\pi\varepsilon_0} \dfrac{\vec{r}-\vec{r}_k}{|\vec{r}-\vec{r}_k|^3}$

$V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\sum\limits_{k=1}^{N}\dfrac{q_k}{|\vec{r}-\vec{r}_k|}$

or:

$\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\int d\tau' \dfrac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}')$

$V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\int d\tau' \dfrac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$

For this purpose we would need the «continuous» charge density associated with a point charge. This can be achieved by using **Dirac's delta function**, writing formally:

Dirac's delta

Point charge q at 0

$\rho_{\text{point charge}} = q\delta^3(\vec{r})$

Point charge q at $\vec{r}_1$

$\rho = q\delta^3(\vec{r}-\vec{r}_1)$

The Dirac's delta function is zero everywhere except at one point, where it has an infinite value: in fact for a point charge we must have $q = \int d\tau \rho(\vec{r}) = \rho(0)d\tau$ where $d\tau \approx 0$

The Dirac's delta is not a conventional function, but it can be defined rigorously as «distribution», that is, as an <u>integrand</u>, using the following definition (f is any function or field):

$\int d\tau\, \delta^3(\vec{r})f(\vec{r}) = f(0)$ As special case with $f(\mathbf{r}) = 1$, this yields : $\int d\tau\, \delta^3(\vec{r}) = \int d\tau\, \delta^3(-\vec{r}) = 1$

(notice that $\delta^3(\vec{r}) = \delta^3(-\vec{r})$)

Dirac's delta & Gauss's law

The Dirac's delta has several interesting properties. The most important for our purposes is the following:

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3} \right) = 4\pi\delta(\vec{r})$$

Proof:

$$\int_{Vol} \left(\vec{\nabla} \cdot \frac{\vec{r}}{r^3} \right) d\tau = \int_S \frac{\vec{r}}{r^3} \cdot d\vec{a} = \int_S \frac{\hat{r}}{r^2} \cdot \hat{r} r^2 d\Omega = \int d\Omega = 4\pi = \int_{Vol} 4\pi\delta(\vec{r}) d\tau$$

Gauss's integral theorem

Since both the initial and final integral yield the same result, they must be equal, which proves the above equality. Note that since $\frac{\vec{r}}{r^3} = -\vec{\nabla}\frac{1}{r}$, we also have that $-\nabla^2\frac{1}{r} = 4\pi\delta^3(\vec{r})$

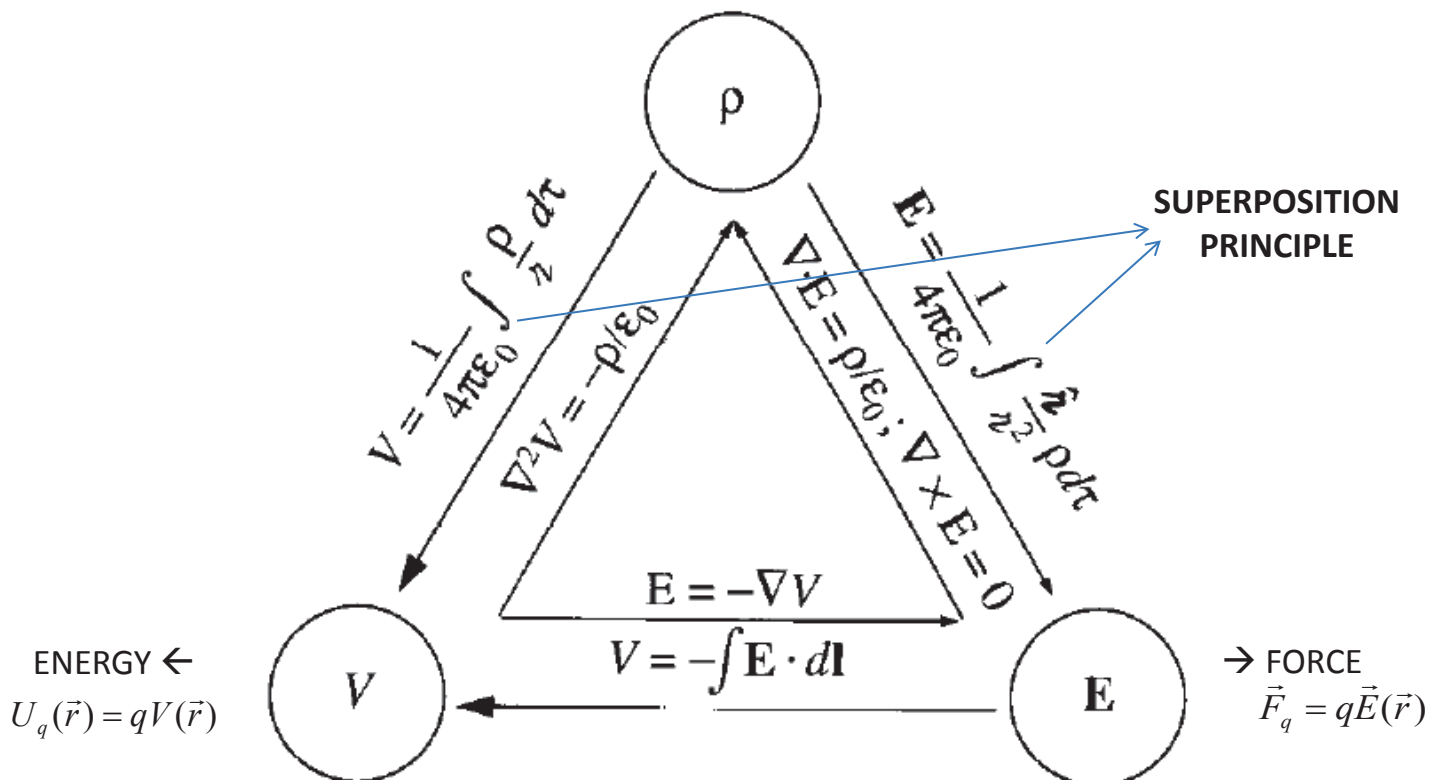### *Proof of Gauss's law :*

$$\vec{E}(\vec{r}) = \int d\tau' \frac{\rho(\vec{r}')}{4\pi\varepsilon_0} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \Rightarrow \vec{\nabla} \cdot \vec{E} = \int d\tau' \frac{\rho(\vec{r}')}{4\pi\varepsilon_0} \vec{\nabla} \cdot \left(\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = \int d\tau' \frac{\rho(\vec{r}')}{4\pi\varepsilon_0} 4\pi\delta(\vec{r} - \vec{r}')$$

$$\Rightarrow \vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\varepsilon_0} \int\int d\tau' \, \rho(\vec{r}')\delta(\vec{r} - \vec{r}') = \frac{\rho(\vec{r})}{\varepsilon_0}$$

This entails, in integral form:
$$\oint_S \vec{E} \cdot d\vec{a} = \int_{Vol} \vec{\nabla} \cdot \vec{E} d\tau = \frac{1}{\varepsilon_0} \int_{Vol} \rho(\vec{r}) d\tau = \frac{Q_{enclosed}}{\varepsilon_0}$$

$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ and $\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$ together imply: $\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$ <u>Poisson's law</u>

Graphical summary of electrostatics

SUPERPOSITION PRINCIPLE

ρ

$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{\imath} d\tau$

$\nabla^2 V = -\rho/\varepsilon_0$

$\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0 \; ; \; \vec{\nabla} \times \vec{E} = 0$

$E = \frac{1}{4\pi\varepsilon_0} \int \frac{\hat{\imath}}{\imath^2} \rho d\tau$

$E = -\nabla V$

$V = -\int \mathbf{E} \cdot d\mathbf{l}$

V

$\mathbf{E}$

ENERGY $\leftarrow$
$U_q(\vec{r}) = qV(\vec{r})$

$\rightarrow$ FORCE
$\vec{F}_q = q\vec{E}(\vec{r})$

$U = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{4\pi\varepsilon_0 |\vec{r}_i - \vec{r}_j|}$

For a continuous distribution $\Rightarrow$

$$U = \frac{1}{2} \int\int \frac{\rho(\vec{r})\rho(\vec{r}')}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|} d\tau d\tau' = \frac{1}{2} \int \rho(\vec{r})V(\vec{r}) d\tau$$
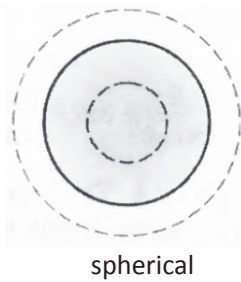
NOTE: $\delta^3(\vec{r})$ cannot be used here
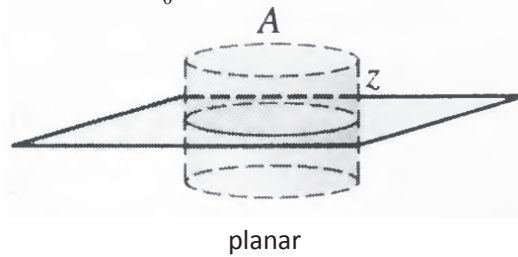
Solving simple electrostatics problems

if the charge density is given:

1) For high symmetry, use **GAUSS's LAW** : $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$
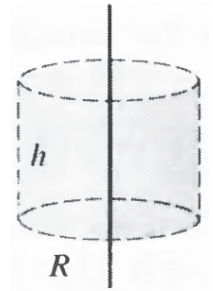
Gaussian surfaces
for 3 symmetries

spherical   planar   cylindrical

2) For low symmetry, **INTEGRATE directly**, either:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d\tau' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} \rho(\vec{r}') \quad \text{or} \quad V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

3) For sum of two or more simple charge distributions → use **SUPERPOSITION PRINCIPLE**

If the potential of conductors is given: (possibly with an external field or charges)
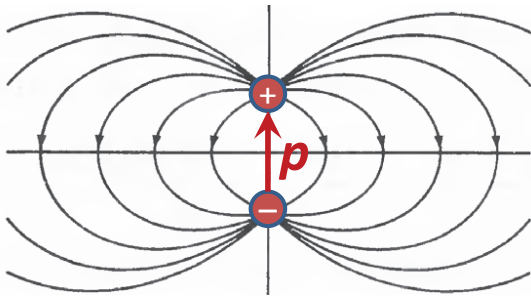
- For high symmetry :

  option A) solve **LAPLACE's EQUATION** directly in region between conductors: $\nabla^2 V = 0$

  option B) guess the direction of E, apply **gauss's law** and use the line integral of E to find the relationship between the charge density and the potential (e.g. capacitance)

- For other cases, option C) try with **IMAGE CHARGES** (see later)   → HOMEWORK

The electric dipole

Consider two equal and opposite charges +q and –q placed a short distance *d* away from one another. The field lines look as in the figure. We define the electric dipole **p** of this set of charges as the vector :

$$\vec{p} = q\,\vec{d}$$, where $\vec{d}$ goes from –q to +q

$[p] = \text{C} \cdot \text{m} \,[\text{SI}]$

$[p] = e \cdot \text{Å} \,[\text{atomic}]$

$[p] = \text{D} \;(\text{Debye})$; 1 D = 0.2082 eÅ
(unit used in (bio)chemistry)

$$\frac{\vec{d}}{2} + \vec{r}_+ = \vec{r} \longrightarrow \vec{r}_- = \vec{r} + \frac{\vec{d}}{2}$$

$$\frac{\vec{d}}{2} + \vec{r} = \vec{r}_- \longrightarrow \vec{r}_+ = \vec{r} - \frac{\vec{d}}{2}$$

The total potential of the 2 charges (r_+ and r_- from figure) is: $V(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_+} - \dfrac{1}{r_-}\right)$

Since: $\dfrac{1}{r^+} = \dfrac{1}{\sqrt{\vec{r}_+ \cdot \vec{r}_+}} = \dfrac{1}{\sqrt{\left(\vec{r}-\frac{\vec{d}}{2}\right)\cdot\left(\vec{r}-\frac{\vec{d}}{2}\right)}} = \dfrac{1}{\sqrt{r^2 - 2\vec{r}\cdot\frac{\vec{d}}{2} + \frac{d^2}{4}}} = \dfrac{1}{\sqrt{r^2 - \vec{r}\cdot\vec{d} + \frac{d^2}{4}}} = \dfrac{1}{r}\dfrac{1}{\sqrt{1 - \frac{\hat{r}\cdot\vec{d}}{r} + \frac{d^2}{4r^2}}}$

and similarly for r_- , we get: $V(\vec{r}) = \dfrac{q}{4\pi\varepsilon_0}\left(\dfrac{1}{r_+} - \dfrac{1}{r_-}\right) = \dfrac{q}{4\pi\varepsilon_0 r}\left(\dfrac{1}{\sqrt{1 - \frac{\hat{r}\cdot\vec{d}}{r} + \frac{d^2}{4r^2}}} - \dfrac{1}{\sqrt{1 + \frac{\hat{r}\cdot\vec{d}}{r} + \frac{d^2}{4r^2}}}\right)$

The point dipole: V and E

The result in the previous slide is correct, but we will use a simpler expression based on the Taylor expansion of both terms of V. Using the expression $(1+\varepsilon)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\varepsilon$, valid for $\varepsilon << 1$, we find:

$$V(\vec{r}) \cong \frac{q}{4\pi\varepsilon_0 r}\left[\left(1+\frac{1}{2}\frac{\hat{r}\cdot\vec{d}}{r}-\frac{1}{2}\frac{d^2}{4r^2}\right)-\left(1-\frac{1}{2}\frac{\hat{r}\cdot\vec{d}}{r}-\frac{1}{2}\frac{d^2}{4r^2}\right)\right]=\frac{q}{4\pi\varepsilon_0 r}\frac{\hat{r}\cdot\vec{d}}{r}=\frac{q\vec{d}\cdot\hat{r}}{4\pi\varepsilon_0 r^2}$$

Since $\vec{p} = q\,\vec{d}$, we have: $\boxed{V(\vec{r}) \approx \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}}$ This is called "point-dipole" potential

If **p** is parallel to the z axis, then, in spherical coordinates: $V_{\substack{\text{point}\\\text{dipole}}} = \frac{p\hat{z}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$

The point-dipole potential represents the (approximate) potential of a very small dipole, or better the potential at distances r >> s. From the point-dipole potential we get the corresponding point-dipole field by taking the gradient of V and changing its sign:

$$\vec{E} = -\vec{\nabla}V = -\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0}\underbrace{\vec{\nabla}\left(\frac{1}{r^3}\right)}_{(-3)r^{-4}\hat{r}}-\frac{1}{4\pi\varepsilon_0 r^3}\underbrace{\vec{\nabla}(\vec{p}\cdot\vec{r})}_{\substack{(\partial_x,\partial_y,\partial_z)(p_x x + p_y y + p_z z) =\\ = (p_x, p_y, p_z)}} \Rightarrow$$

$$\Rightarrow \boxed{\vec{E}(\vec{r})_{\substack{\text{point}\\\text{dipole}}} = \frac{3\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0}\frac{\vec{r}}{r^5}-\frac{\vec{p}}{4\pi\varepsilon_0 r^3}}$$ If **p** is parallel to z, then $\vec{E}_{\substack{\text{point}\\\text{dipole}}} = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos\theta\,\hat{r}-\hat{z})$
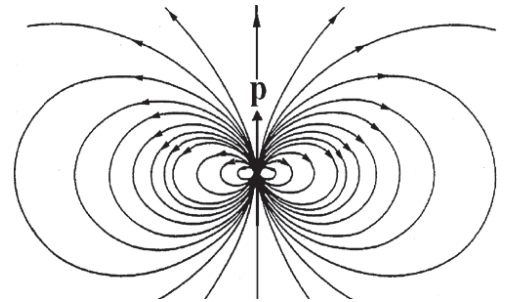
Real dipoles: permanent vs induced

Types of real electric dipoles:

Permanent dipoles (**p** is fixed)

1) Permanent dipoles: polar molecules in which the center of positive charge does not coincide with that of negative charge (CO, H_2O, Na^+Cl^-, …)


(monoatomic molecules (noble gases: He, Ne, Ar, …), and symmetric molecules such as H_2, N_2, O_2, …, CO_2, methane CH_4, benzene C_6H_6, etc., do NOT possess a permanent electric dipole moment. Compare CO_2 with H_2O: molecular structure, not formula, counts!)

O = C = O

p = 0
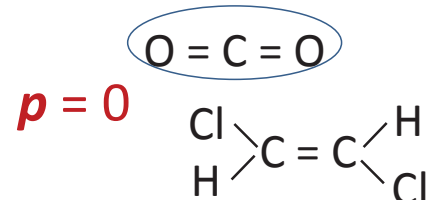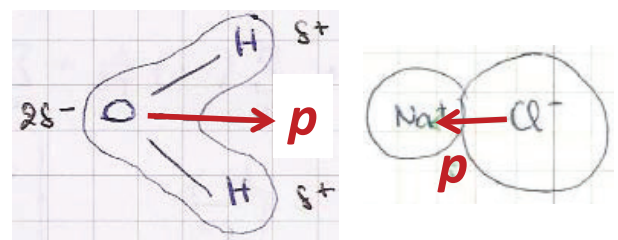
2) Induced dipoles: apolar atoms and molecules under and applied field become «polarized», that is, the center of positive charge (nucleus for an atom) moves away from the field and the center of negative charge (center of the electron cloud for an atom) moves towards the field: $\vec{p} = \alpha\vec{E}$

Induced dipole (**p** depends on E)

$$\vec{p} = \alpha\vec{E}$$
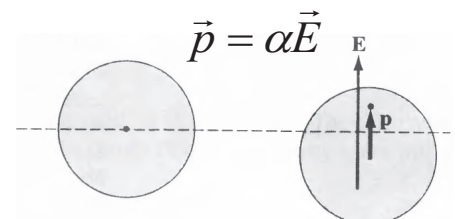
Apolar atom/molecule in an external E

→ the electronic cloud is displaced from nucleus (induced polarization)

The net force on the nucleus is the sum of the force due to the external field and the force due to the electron cloud:
at equilibrium $E_{el} = E_{ext}$
Assuming that the cloud is not deformed by the applied field, the nucleus is at a distance d from the center of the cloud and the electronic force is calculated with Gauss' law:
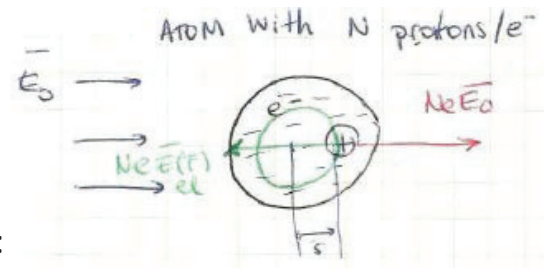
$$-E_{el}(s)4\pi d^2 = \frac{Q_{ror}(d)}{\varepsilon_0} = \frac{\rho_{el}\frac{4}{3}\pi d^3}{\varepsilon_0} \implies \vec{E}_{el}(\vec{d}) = -\frac{\rho_{el}\vec{d}}{3\varepsilon_0} = \vec{E}_0$$

$$\implies \vec{d} = -\frac{3\varepsilon_0}{\rho}\vec{E}_0\frac{Vol}{Vol} = -\frac{3\varepsilon_0 Vol}{Ne}\vec{E}_0 \implies \vec{p}\big(=-Ne\vec{d}\big) = \alpha\vec{E} \quad \text{with:} \quad \alpha = 3\varepsilon_0 Vol$$

Vol = atomic volume

Note: here N is really the number of valence (outermost) e^-, since they are more easily polarized But one should really do a quantum mechanical calculation ! In quantum mechanics the dipole moment is the operator $-e\mathbf{r}$ and $\vec{p} = \langle\psi|-e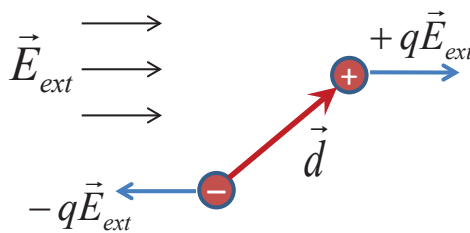\vec{r}|\psi\rangle$. For the hydrogen atom under an applied field, for example : $p = \frac{9}{2}(4\pi\varepsilon_0 a_B^3)E_0\hat{k}$

$\frac{\alpha}{\varepsilon_0} \sim Vol$ Polarizability α/ε_0 (in Å^3) for some atoms and molecules

He	2.5	H$_2$	10	H$_2$O	19
Ne	5.0	O$_2$	19	CH$_4$	33
Ar	20	N$_2$	22	CO$_2$	33
Kr	31	CO	24	CCl$_4$	127
Xe	51				

Polar atom/molecule in an external E

The permanent dipole moment of the atom/molecule orients parallel to the applied field

$$\vec{F}_{tot} = +q\vec{E}_{ext} - q\vec{E}_{ext} = 0 \quad \text{(if } \mathbf{E}_0 \text{ is uniform!)}$$

$$\vec{\Gamma}_{tot} = +q\vec{d}\times\vec{E}_{ext} = \vec{p}\times\vec{E}_{ext}$$

$\vec{\Gamma}$ for a force par ($\mathbf{F}_{tot} = 0$) does not depend on the reference; we choose $-q$)

Energy of a permanent dipole in an applied field:

$$U_{\substack{rigid \\ permanent \\ dipole}} = +qV_{ext}(\vec{r}+\vec{d}) - qV_{ext}(\vec{r}) \cong +q\big[(V_{ext}(\vec{r}) + \vec{\nabla}V_{ext}(\vec{r})\cdot\vec{d}\big] - qV_{ext}(\vec{r}) \approx q\vec{d}\cdot\vec{\nabla}V_{ext} = -\vec{p}\cdot\vec{E}_{ext}$$

$$\implies \boxed{U_{\substack{permanent \\ point\ dipole}} = -\vec{p}\cdot\vec{E}_{ext}}$$

If the applied field $\mathbf{E}_0$ is not uniform but depends on position, $\mathbf{E} = \mathbf{E(r)}$, the net force is non-zero:

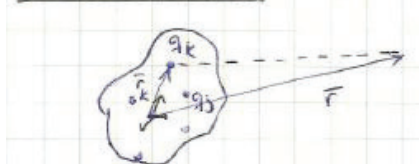$$\vec{F}(\vec{r}) = -\vec{\nabla}U = \vec{\nabla}\big[\vec{p}\cdot\vec{E}(\vec{r})\big]$$

Permanent dipole moment **p** of some molecules (in Debyes and eÅ) ; 1 D = 0.2082 eÅ

molecule	CO	H$_2$O	NH$_3$	HF	HCl	NaCl	CH$_3$Cl	CH$_2$Cl$_2$	CHCl$_3$	C$_3$H$_7$Cl
p (D)	0.122	1.85	1.42	1.75	1.04	9.61	1.87	1.56	1.15	2.06
p (eÅ)	0.025	0.4	0.3	0.36	0.22	2	0.4	0.32	0.24	0.43

Large polymers/biomolecules → hundreds of Debyes!   **Q: how much is p for CH$_4$? And for CCl$_4$?**

Multipole expansion

SET OF CHARGES

$$V_{tot} = \frac{1}{4\pi\varepsilon_0} \sum_{k=1}^{N} \frac{q_k}{|\vec{r} - \vec{r}_k|}$$

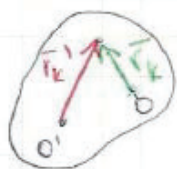$$(1+\varepsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \ldots..$$

$$\frac{1}{|\vec{r}-\vec{r}_k|} = \frac{1}{\sqrt{(\vec{r}-\vec{r}_k)\cdot(\vec{r}-\vec{r}_k)}} = \frac{1}{\sqrt{r^2 + r_k^2 - 2\vec{r}\cdot\vec{r}_k}} = \frac{1}{r}\left(1 - \frac{2\hat{r}\cdot\vec{r}_k}{r} + \frac{r_k^2}{r^2}\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 - \frac{1}{2}\left(\frac{r_k^2}{r^2} - 2\frac{\hat{r}\cdot\vec{r}_k}{r}\right) + \frac{3}{8}(\ )^2\right)$$

$$\Rightarrow V_{tot} = \frac{1}{4\pi\varepsilon_0}\sum_k \frac{q_k}{|\vec{r}-\vec{r}_k|} \approx \frac{1}{4\pi\varepsilon_0 r}\sum_k q_k + \frac{1}{4\pi\varepsilon_0 r^2}\sum_k q_k \hat{r}\cdot\vec{r}_k + \frac{1}{r^3}\sum_k ( ) + \ldots.$$

$$\Rightarrow \boxed{V_{tot}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left\{\left(\underbrace{\sum_k q_k}_{=Q_{TOT}}\right)\frac{1}{r} + \left(\underbrace{\sum_k q_k \vec{r}_k}_{=\vec{p}_{TOT}}\right)\cdot\frac{\hat{r}}{r^2} + \ldots.. \frac{1}{r^3} + \ldots. \frac{1}{r^4} + \ldots\right\}}$$

multipole expansion for V
(integer powers of 1/r)

What happens if we choose another point as origin?

$$\vec{p}' = \sum_k q_k \vec{r}_k' = \sum_k q_k(\vec{r}_k + \overrightarrow{OO'}) =$$

$$= \sum_k q_k\vec{r}_k + \left(\sum_k q_k\right)\overrightarrow{OO'} = \vec{p} + Q_{TOT}\overrightarrow{OO'}$$

$$\Rightarrow$$

for an overall neutral system:

$$\boxed{V \approx \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}}$$

$$\vec{p} \text{ is ABSOLUTE}$$

p: key electric parameter neutral molecules

$\vec{p}$ determines
- the field produced by the molecule (in a condensed-matter context)
- the torque on the molecule in an applied field
- the orientational energy of the molecule in an applied field

In a <u>homonuclear</u> molecule, **p** = 0 . Examples: He, Ar, H_2, O_2, C_{60}

In <u>centrosymmetric</u> molecules, **p** = 0 . Examples: CH_4, CO_2, benzene, $C_{60}H_{60}$

In a <u>heteronuclear</u> molecule (e.g. a diatomic heteronuclear molecule), p is largest when the difference between the electron affinity of the two elements is largest. For example, in halogens the electron affinity diminishes with atomic number, since the extra electron is further and further away from the nucleus; as a result, **p** for e.g. hydrogen halides decreases:

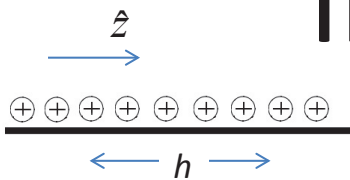| halide	HF	HCl	HBr	HI
$p\cdot10^{30}$ C·m	6.7	3.5	2.6	1.3

In larger organic molecules, as well as in polymers, DNA, proteins, **p** can be very roughly estimated as the vector sum of the dipole moment of all polar side groups:

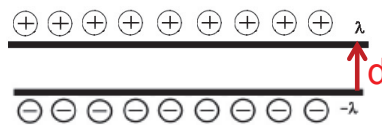| group/bond	C-H	O-H	N-H	$N-O_2$	C-N	C≡N	C-O	C-Cl	C-F	C=O
$p\cdot10^{30}$ C·m	1.33	5.27	5.53	13.2	2.03	13.1	3.73	6.83	6.1	8.7

Ex.: p of alcohols (Debyes): methanol 1.70 ; ethanol 1.69 ; 1-propanol 1.68; isopropanol 1.66
BUT, it doesn't work so well for conjugated molecules: phenol 1.22 D (1 D $\simeq$ 3.336 10^{30} C·m)

The line dipole

$\hat{z}$

line charge:

$\oplus \oplus \oplus \oplus \oplus \oplus \oplus \oplus \oplus$ λ

$\longleftarrow h \longrightarrow$

$$E(s)\cdot 2\pi\, s\, h = \frac{\lambda h}{\varepsilon_0} \Rightarrow \vec{E}(\vec{s}) = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s} \;; \quad V(\vec{s}) = -\frac{\lambda}{2\pi\varepsilon_0}\ln s$$

Line dipole:

$\oplus \oplus \oplus \oplus \oplus \oplus \oplus \oplus \oplus$ λ

$\ominus \ominus \ominus \ominus \ominus \ominus \ominus \ominus \ominus$ $-\lambda$

↑d

We define: $\vec{\wp} = \lambda \vec{d}$ ("dipole density")

$$V_{tot}(\vec{s}) = V_{+\lambda}(\vec{s}_+) + V_{-\lambda}(\vec{s}_-) = -\frac{\lambda\ln s_+}{2\pi\varepsilon_0} + \frac{\lambda\ln s_-}{2\pi\varepsilon_0} = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{s_-}{s_+}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{s_-^2}{s_+^2}\right)$$

← since Log(a²) = 2Log(a)

With $\begin{cases}\dfrac{\vec{d}}{2} + \vec{s}_+ = \vec{s} \\[4pt] \dfrac{\vec{d}}{2} + \vec{s} = \vec{s}_-\end{cases}$, we get : $V_{tot}(\vec{R}) = \dfrac{\lambda}{4\pi\varepsilon_0}\ln\left(\dfrac{s^2 + 2\vec{s}\cdot\dfrac{\vec{d}}{2} + \dfrac{d^2}{4}}{s^2 - 2\vec{s}\cdot\dfrac{\vec{d}}{2} + \dfrac{d^2}{4}}\right) = \dfrac{\lambda}{4\pi\varepsilon_0}\ln\left(\dfrac{1 + \dfrac{\hat{s}\cdot\vec{d}}{s} + \dfrac{d^2}{4s^2}}{1 - \dfrac{\hat{s}\cdot\vec{d}}{s} + \dfrac{d^2}{4s^2}}\right)$

Using the Taylor expansion: $\ln(1+\varepsilon) \approx \varepsilon \Rightarrow V_{tot} = \dfrac{\lambda}{4\pi\varepsilon_0}\left[\ln(1+\varepsilon) - \ln(1+\varepsilon')\right] \approx \dfrac{\lambda}{4\pi\varepsilon_0}(\varepsilon - \varepsilon')$

$$\Rightarrow V_{tot} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{\hat{s}\cdot\vec{d}}{s} + \frac{d^2}{4s^2} - \left(-\frac{\hat{s}\cdot\vec{d}}{s} + \frac{d^2}{4s^2}\right)\right] = \frac{\lambda\hat{s}\cdot\vec{d}}{2\pi\varepsilon_0 s} = \frac{\lambda\vec{d}\cdot\vec{s}}{2\pi\varepsilon_0 s^2} \Rightarrow \boxed{V_{\substack{line\\dipole}} = \frac{\vec{\wp}\cdot\vec{s}}{2\pi\varepsilon_0 s^2} = \frac{\vec{\wp}\cdot\hat{s}}{2\pi\varepsilon_0 s}}$$

The line-dipole field is then: $\boxed{\vec{E}_{\substack{line\\dipole}} = \frac{1}{2\pi\varepsilon_0}\left(\frac{2\vec{\wp}\cdot\vec{s}}{s^4}\vec{s} - \frac{\vec{\wp}}{s^2}\right)} = -\vec{\nabla}V_{\substack{line\\dipole}}$ (gradient calculated in cylindrical coordinates)

If $\vec{\wp} = \wp\hat{x}$, then: $V_{\substack{line\\dipole}} = \dfrac{\wp\hat{x}\cdot\hat{s}}{2\pi\varepsilon_0 s} = \dfrac{\wp\cos\theta}{2\pi\varepsilon_0 s}$ and $\vec{E}_{\substack{line\\dipole}} = \dfrac{\wp}{2\pi\varepsilon_0 s^2}\left(2\cos\theta\,\hat{s} - \hat{x}\right)$

Summary

	V	E
Point charge	1/r	~1/r²
Point dipole	cosθ/r²	~1/r³
Line charge	Ln(s)	~1/s
Line dipole	cosθ/s	~1/s²
Plane charge	z	*constant*
Plane dipole	*constant*	0

Question for the class: why is the electrostatic field of a plane dipole zero?

What about volume dipoles? → see next slide: a "sphere dipole"
→ see TOPIC 2 !!

Multipole expansion for a DELIMITED charge distribution:

Q_{tot} $\vec{P}_{tot}$

$$V(\vec{r}) = \frac{1}{r}\frac{\overbrace{\left(\sum q_k\right)}}{4\pi\varepsilon_0} + \frac{1}{r^2}\frac{\overbrace{\left(\sum q_k\vec{r}_k\right)\cdot\hat{r}}}{4\pi\varepsilon_0} + \frac{1}{r^3}(\ ) + \frac{1}{r^4}(\ ) + \ldots$$
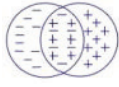
Sphere dipole (plasma model for the point dipole)

uniformly
charged
sphere

$$\vec{E}(r)4\pi r^2 = \frac{\rho \frac{4}{3}\pi r^3}{\varepsilon_0} \Rightarrow \vec{E}(r) = \frac{\rho \vec{r}}{3\varepsilon_0}$$
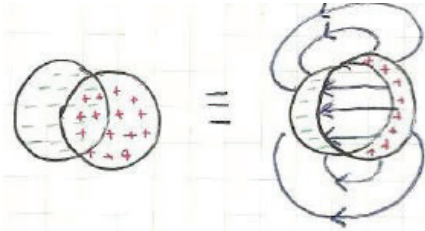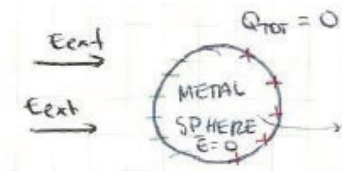
point dipole

field inside plasma:

$$\vec{E}_{tot} = \vec{E}_+ + \vec{E}_- = \frac{\rho \vec{r}_+}{3\varepsilon_0} + \left(-\frac{\rho \vec{r}_-}{3\varepsilon_0}\right) = \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = -\frac{\rho \vec{d}}{3\varepsilon_0}$$

field outside plasma → point dipole field
(use superposition principle to see why)

Example: metal sphere in applied uniform field

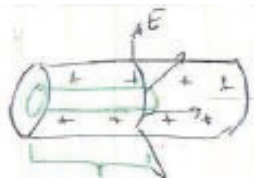Inside the conducting sphere, we have $\vec{E}_{tot} = \vec{E}_{ext} + \vec{E}_{response} = 0$

$$\vec{E}_{response} = -\vec{E}_{ext} = -\vec{E}_0 \Rightarrow \vec{E}_{response} = -\frac{\rho \vec{d}}{3\varepsilon_0}$$

$$\vec{E}_{response} = -\vec{E}_{ext} = -\frac{\rho \vec{d}}{3\varepsilon_0} \Rightarrow \rho\vec{d} = 3\varepsilon_0 \vec{E}_{ext} \Rightarrow \vec{p} = \rho Vol\,\vec{d} = 3\varepsilon_0 Vol\,\vec{E}_{ext}$$
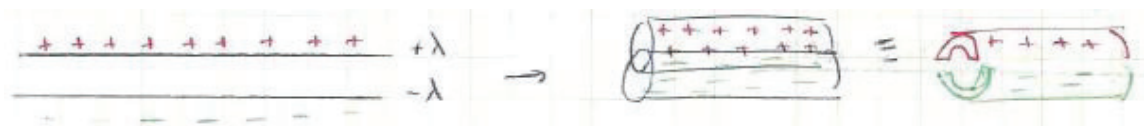
Plasma model of the line dipole

uniformly
charged
cylinder

$$E(s)2\pi sh = \rho\frac{\pi s^2 h}{\varepsilon_0} \Rightarrow \vec{E}(\vec{s}) = \frac{\rho \vec{s}}{2\varepsilon_0}$$
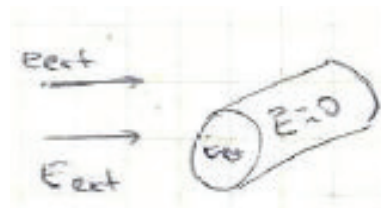
line dipole

field inside plasma → $\vec{E}_{tot} = \vec{E}_+(\vec{s}_+) + \vec{E}_-(\vec{s}_-) = \rho\frac{\vec{s}_+}{2\varepsilon_0} - \rho\frac{\vec{s}_-}{2\varepsilon_0} = -\frac{\rho}{2\varepsilon_0}\vec{d}$ (we used $\vec{d} + \vec{s}_+ = \vec{s}_-$)

field outside plasma → line dipole field (use superposition principle to see why)

Example: metal wire in applied uniform field

$$\vec{E}_{tot} = \vec{E}_{ext} + \vec{E}_{resp} = \vec{E}_{ext} - \frac{\rho \vec{d}}{2\varepsilon_0} = 0 \quad \Rightarrow$$

$$\vec{E}_{ext} = \frac{\rho \vec{d}}{2\varepsilon_0} = \frac{\lambda \vec{d}}{2\varepsilon_0 Area_{cross\ section}} = \frac{\vec{\wp}}{2\varepsilon_0 Area_{cross\ section}} \Rightarrow \vec{\wp}_{induced\ line\ dipole} = 2\varepsilon_0 Area_{cross\ section}\,\vec{E}_{ext}$$

here we used the fact that $\lambda h = Q = \rho Vol = \rho\pi a^2 h \Rightarrow \rho = \frac{\lambda}{\pi a^2} = \frac{\lambda}{Area_{cross\ section}}$

We will solve again the last 2 examples in a more formal way using the method of *image charges*

Electrostatic properties of conductors

Fundamental electrostatic property of good conductors (for examlpe metals):

In a conductor with no flow of current (in "electrostatic equilibrium"):

$$E_{inside} = 0$$

This is so because if $E \neq 0$, free charges would move, accelerated by the field, so there would not be equilibrium. Free charges accumulate in such a way as to make $E_{inside} = 0$

Notice that $E_{inside} = 0$ only in electrostatics; if there is a current running through the conductor, then E is different from zero (it is in fact given by Ohm's law, see topic 4)

Consequences of fundamental property:

- applying Gauss's law → $\boldsymbol{\rho_{inside}} = \boldsymbol{0}$: net charge = 0 inside a conductor ; the charge resides entirely on the surface (where the surface charge density $\boldsymbol{\sigma \neq 0}$, in general)

- integrating E → $\boldsymbol{V = const}$: surface & volume of a conductor are equipotential; the surface is an equipotential surface; <u>no E-field lines start at end on same conductor</u>

- (boundary condition) → $\boldsymbol{E_{tangential} = 0}$ just outside conductor. Hence just outside the conductor, E is normal to the surface

- (boundary condition). The field just outside a conductor is proportional to the local surface charge density and orthogonal to the surface:

→ if **r** belongs to the conductor's surface, $\boldsymbol{E_n(r) = \sigma(r)/\varepsilon_0}$

What does $E_{inside} = \rho_{inside} = 0$ really mean?

A metal is actually made of ions and electrons; how can the charge density and the electric field be zero near a point-like charge like an atomic nucleus or an electron?? When we say E = 0 or **ρ** = 0, we mean an average, "macroscopic" quantities, namely the field or charge densities AVERAGED over distances that are large compared to atomic ones (~ 1Å). We denote the microscopic volume charge density by η and the microscopic (full-detail) field created by it as **e**. They fulfill: $\boldsymbol{\nabla \cdot e} = \eta/\epsilon_0, \quad \boldsymbol{\nabla \times e} = 0$

In general, the spatial average of a function $F(\mathbf{r})$ or $F(\mathbf{r}, t)$ can be defined as:

$\langle F(\mathbf{x}, t) \rangle = \int d^3x' \; f(\mathbf{x}')F(\mathbf{x} - \mathbf{x}', t)$, where the sampling

function f(**r**) is a function that is non-zero in a neighborhood of **r** = 0, and normalized to 1 over all space. Possible examples are:

$$f(\mathbf{x}) = \begin{cases} \dfrac{3}{4\pi R^3}, & r < R \\ 0, & r > R \end{cases}$$

$$f(\mathbf{x}) = (\pi R^2)^{-3/2} e^{-r^2/R^2}$$

The <u>macroscopic</u> charge densities and field are defined as: $\rho(\vec{r}) = \langle \eta(\vec{r}) \rangle \; ; \vec{E}(\vec{r}) = \langle e(\vec{r}) \rangle$

It can be shown that applying the averaging on the microscopic field equations, they remain valid for the average fields, so that indeed $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ and $\vec{\nabla} \times \vec{E} = 0$

Question for the class: apart from metals, which other systems fulfill $E_{inside} = 0$?
In other words, which systems have charges that are free to move around?
(*hint*: think about the 3 or 4 possible states of matter that you know of..)
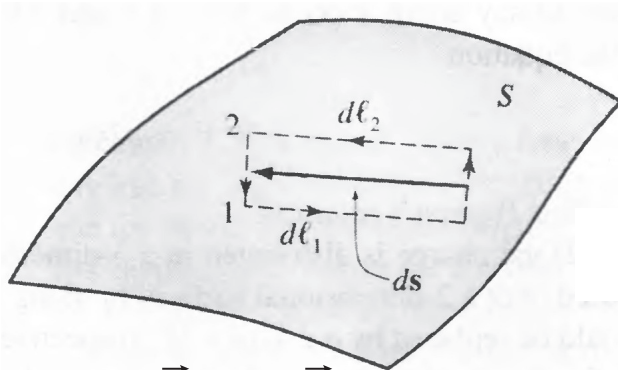
Boundary conditions with conductors

What we did in the last lines of the previous slide is apply so-called **boundary conditions**, which are general electrostatic relations, to the case of an interface between a conductor & vacuum:

$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow$$

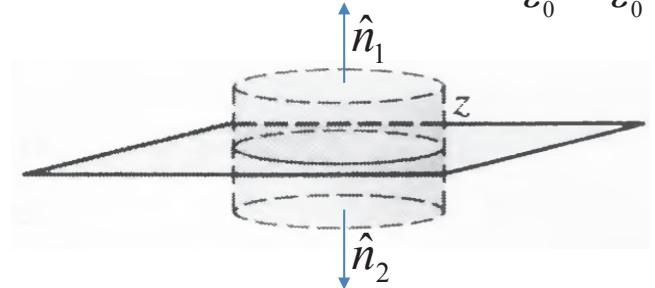$$E_{t2}d\ell_2 + E_{t1}d\ell_1 = (E_{t2} - E_{t1})|d\ell| = 0$$

$$\Rightarrow \vec{E}_{t1} = \vec{E}_{t2}$$

In electrostatics, $\vec{E}_{t1} = \vec{E}_{t,\text{inside}} = 0$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow$$

$$\Phi(\vec{E}) = \vec{E}_1 \cdot \hat{n}_1 A + \vec{E}_2 \cdot \hat{n}_2 A = \frac{Q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow \vec{E}_1 \cdot \hat{n}_1 + \vec{E}_2 \cdot \hat{n}_2 = \sigma/\varepsilon_0$$

In electrostatics, $\vec{E}_1 = \vec{E}_{\text{inside}} = 0$

hence: $E_{t,\text{just outside}} = 0$ hence: $\vec{E}_{\text{just outside}} \cdot \hat{n} = \left|\vec{E}_{\text{just outside}}\right| = \sigma/\varepsilon_0$

$$\Rightarrow \sigma = \varepsilon_0 \vec{E}_{\text{just outside}} \cdot \hat{n} = -\varepsilon_0 \vec{\nabla} V \cdot \hat{n}\Big|_{\text{just outside}}$$

Capacitance and electrostatic energy

Capacitance

The total surface charge on a single conductor far from other charges and conductors turns out to be directly proportional to the voltage we apply to it (we will show this in general later). It is then useful to define the *capacitance* of a conductor, as

$$C = Q/V$$

Here Q is the total surface charge ON THE CONDUCTOR.
If they carry equal and opposite charges, we can also define a capacitance for a set of two conductors (*capacitor*)

SI unit of capacitance : C/V = F (Farad)

Energy stored on a single conductor:

$$U = \frac{1}{2}\int \sigma(\vec{r})V(\vec{r})da \quad, \quad V(\vec{r}) = const = V \quad \Rightarrow U = \frac{1}{2}V\int \sigma(\vec{r})da = \frac{1}{2}VQ$$

$$C = \frac{Q}{V} \quad \Rightarrow \quad U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$

Laplace's law & Uniqueness theorem

If we have a set of conductors and for each of them we specify the potential V or the charge density $\sigma = \varepsilon_0 E_n = -\varepsilon_0 \vec{\nabla} V \cdot \hat{n}$, then the solution to Laplace's equation $\nabla^2 V = 0$ in the region between conductors is unique.

Proof:

Call S the (grey) area outside the conductors, and δS its boundary. Let's suppose that there are two solutions $V_A(\vec{r})$ and $V_B(\vec{r})$. Consider the field $\vec{C}$ defined as $\vec{C} = (V_B - V_A)\vec{\nabla}(V_B - V_A)$

Applying Gauss's theorem to $\vec{C}$ we get $\int_S d\tau\, \vec{\nabla} \cdot \vec{C} = \oint_{\delta S} \vec{C} \cdot \hat{n}\, da = 0$

The last equality follows from the fact that at the boundary δS
it is either $V_A = V_B$ or $\sigma_A = \sigma_B \Rightarrow (\vec{\nabla} V_A) \cdot \hat{n} = (\vec{\nabla} V_B) \cdot \hat{n}$

In both cases the flux of $\vec{C}$ through δS is zero
The divergence of $\vec{C}$ is equal to (product rule) :

$$\vec{\nabla} \cdot \vec{C} = \vec{\nabla} \cdot \left[(V_B - V_A)\vec{\nabla}(V_B - V_A)\right] = \vec{\nabla}(V_B - V_A) \cdot \vec{\nabla}(V_B - V_A) + (V_B - V_A)\vec{\nabla} \cdot \vec{\nabla}(V_B - V_A) =$$

$$= \left|\vec{\nabla}(V_B - V_A)\right|^2 + (V_B - V_A)\nabla^2(V_B - V_A) = \left|\vec{\nabla}(V_B - V_A)\right|^2$$

(the last step follows since V_A and V_B are solutions to Laplace's equation)

Hence we find $\int_S d\tau\, \vec{\nabla} \cdot \vec{C} = \int_S d\tau \left|\vec{\nabla}(V_B - V_A)\right|^2 = 0$. Since the integrand is everywhere ≥ 0, the integral is zero only if at all points one has $\vec{\nabla}(V_B - V_A) = 0$. But this implies that $\vec{E}_A = \vec{E}_B$, that is, the two solutions are in fact the same.
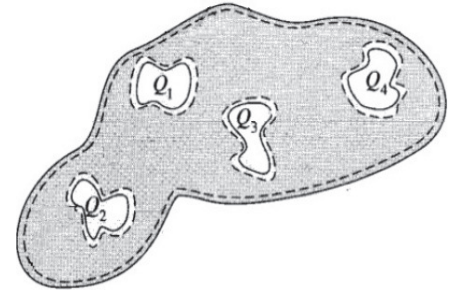
Application: capacitance of a conductor

If a conductor is charged or a voltage is applied to it, it will exhibit a surface charge density and total surface charge given by

$$\sigma_0 = \varepsilon_0 \vec{E}(surface) \cdot \hat{n} \Rightarrow Q_0 = \int_{surface} \sigma_0 da$$

Suppose that the potential of the conductor is v_0, and that there is no other charge/conductor nearby. The potential infinitely far will be zero. The potential inside is constant and the field inside is zero. The potential outside V(**r**) is the solution to Laplace's equation $\nabla^2 V = 0$ with the above-mentioned boundary conditions. Suppose we change the potential of the conductor to v_1 The potential everywhere V'(**r**) must be solution to Laplace's equation and satisfy the new boundary condition. If we knew V(**r**) before, it's easy to guess a new potential V'(**r**) that does the job: it is simply $V'(\vec{r}) = \dfrac{v_1}{v_0} V(\vec{r})$

Since the solution is unique, this must be it. The new field will be $\vec{E}'(\vec{r}) = \dfrac{v_1}{v_0} E(\vec{r})$ and the

corresponding surface charge density and total charge: $\sigma'(\vec{r}) = \dfrac{v_1}{v_0}\sigma_0(\vec{r}) \Rightarrow Q_1 = \dfrac{v_1}{v_0} Q_0$

If instead of requiring the new voltage to be v_1 we specify the new total charge to be Q_1, we already have a solution: it is the one we just found, with the same voltage v_1
We see therefore that the form (dependence on r) of the surface charge distribution is unique for the given conductor, it only rescales with the total charge. We see also that Q is linearly proportional to v, that is, that Q/v is a constant C. This constant is called *capacitance*

Application 2: capacitance of 2 conductors

Suppose we take 2 conductors, separated from everything else, and charge them with equal but opposite charges +Q and –Q. The potentials of the two conductors will be V_A and V_B, respectively (with $V_A > V_B$). Provided that the absolute value $|Q| = Q$ of the charge is the same, we define the capacitance of the system as:

$$C = \frac{|Q|}{\Delta V} = \frac{Q}{V_A - V_B}$$
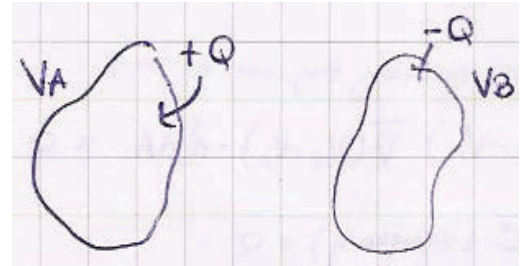
Here $\Delta V = V_A - V_B > 0$ is the potential difference between the two conductors

The condition that the total charge on the two conductors is the same (apart from the sign) is easy to fulfill. For example, if a potential difference is established between two initially uncharged conductors by connecting them to opposite electrodes of a battery, then what the battery does is transport electrons from one conductor to the other. In other words, the two conductors will have at the end of the charging process equal but opposite charges.

For a given value of $|Q|$, the corresponding potential V(**r**) everywhere will be the solution of Poisson's equation with the appropriate boundary conditions, which fixes the value of the surface charge density σ at all points on the surface of the conductor. If we now rescale the function σ by a constant factor, the value of $|Q|$ and of the potential will scale according to the same factor → **the capacitance C of a system of two conductors having equal and opposite charge is** really a **constant**, and we can treat it as such when solving electrostatic problems!

Capacitance: exact calculation vs estimate

- **Calculation of C for the a single conducting slab, sphere or cylinder**   (→ exercise classes)
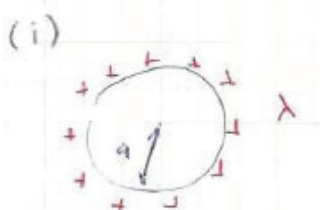- **Calculation of C for the planar, spherical and cylindrical capacitors**

Two possible approaches in both cases: guess the symmetry of E ; else, solve Laplace's equation

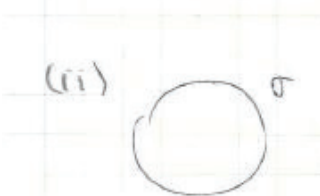- **Approximate estimate**: example: thin disk of radius b
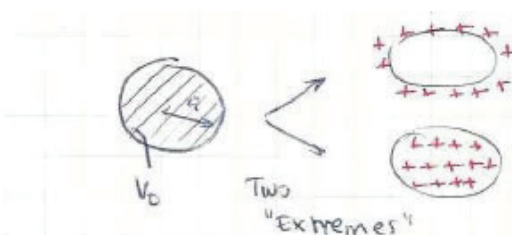
$$\lambda = \text{constant} \quad (i)$$

$$\sigma = \text{constant} \quad (ii)$$

(i)

$$V_{center} = \int \frac{\lambda d\ell}{4\pi\varepsilon_0 b} = \frac{1}{4\pi\varepsilon_0 b}\int \lambda d\ell = \frac{Q_{tot}}{4\pi\varepsilon_0 b}$$

$$Q_{tot} = \lambda 2\pi b \Rightarrow C = \frac{Q_{tot}}{V_{center}} = 4\pi\varepsilon_0 b$$

$$\Rightarrow C \approx 3\pi\varepsilon_0 b$$

(ii)

$$V_{center} = \int_0^b \frac{\sigma 2\pi R dR}{4\pi\varepsilon_0 R} = \frac{\sigma}{2\varepsilon_0}$$

$$Q = \sigma\pi b^2 \Rightarrow C = 2\pi\varepsilon_0 b$$

Image charge method for conductors

If we have charges, dipoles, etc. outside a conductor, these external charges create, by themselves, a potential that is not uniform in the space occupied by the conductor. This potential, or field, induces a peculiar charge density on the surface of the conductor which together with the external charges generates a total potential that is constant (say, equal to V_0) over the whole volume of the conductor, and that satisfies Laplace's equation everywhere outside it except at the points where the external charges are located. The problem is to find the surface charge distribution that does the job.

The image charge method is based on the uniqueness theorem, which states that there is a unique solution to Laplace's equation with complete boundary conditions.

The image charge method for conductors works this way: starting from the non-uniform potential created by external charges, we try to guess a fictitious, simple set of charges or dipoles that, if placed at specific position inside the conductor, render the surface of the conductor an equipotential surface, of given potential V_0. If we succeed to do so, we're almost done: the set of real external charges plus fictitious (image) charges together create a potential that fulfills the following requirements: it is equal to V_0 on the conductor surface, and zero at infinity; it is a solution of Laplace's equation everywhere outside the metal except at the points where the real external charges are located. From the uniqueness theorem, we then know that this potential is the (unique) solution to the initial problem.

Since we now know the potential everywhere outside the conductor, we can calculate the field everywhere outside; we can then use the boundary condition on the normal component of **E** to find the surface charge density induced on the conductor.

Simple example of image charge I

A point charge q lies a distance z_0 above a semi-infinite (grounded) metal half-space. It induces a charge density on the planar surface of the metal. What is V in the region above the plane?
Answer: the induced surface charge must compensate the effect of the external charge: the metal surface should be equipotential, but due to charge q it is not. **Is there a simple way that we can make the surface potential to be everywhere zero?**
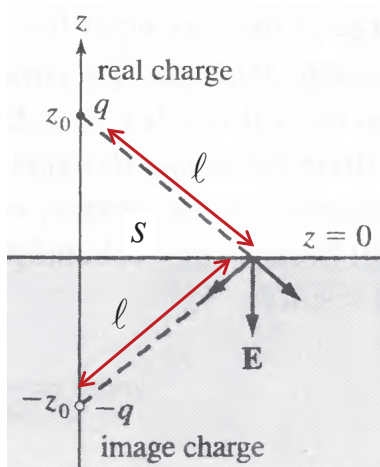
Yes: if we put a charge $-q$ a distance $-z_0$ below the plane, then the surface is halfway between two equal and opposite charges, and it is straightforward to see that on such midplane $V_{TOT} = 0$:

$$V_{TOT} = V_{+q} + V_{-q} = \frac{q}{4\pi\varepsilon_0 \ell} + \frac{-q}{4\pi\varepsilon_0 \ell} = 0 \implies V_{TOT} = 0 \text{ for z = 0}$$

The potential $V_{TOT} = V_{+q} + V_{-q}$ satisfies all the conditions of the uniqueness theorem in the region outside the metal:
- Outside the metal, V_{TOT} satisfies Laplace's equation $\nabla^2 V_{TOT} = 0$ everywhere except at the position of the external charge;
- $V_{TOT} = 0$ at the metal surface;
- $V_{TOT} \to 0$ at a point infinitely far from the external charge.

Since the solution for V is unique, the potential $V_{TOT} = V_{+q} + V_{-q}$ must be the true potential outside the metal for this electrostatic problem. Of course, we cannot put a free charge at rest in the middle of a conductor: the negative charge at (0,0,–d) is a fictitious IMAGE charge, a useful tool to determine V and E outside the metal. These fields (V, E) are really produced by the external charge and by the **induced surface charge** on the conductor, which is distributed in such a way as to give, outside the metal, the same field of a (fictitious) point charge!

Simple example of image charge II

real charge

z_0 q

S

$z = 0$

E

$-z_0$ $-q$

image charge

$z = 0$
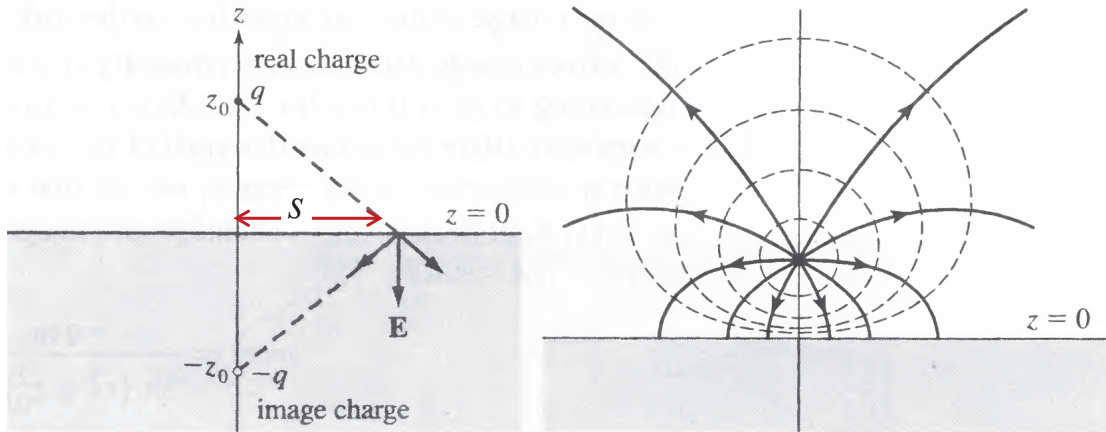
The image charge is just a useful theoretical tool: there can be no charge inside a conductor in electrostatics. What is then the real induced surface density on the plane? To calculate it, first notice that the total electric field at the conductor's surface must be normal to it. Since such field can be obtained as the sum of the fields of two point charges, this is easily verified. In fact, we see that such field is simply $E_n = 2(E_{ext})_z$, where $(E_{ext})_z$ is the vertical component of the field due to the external charge at the metal surface.
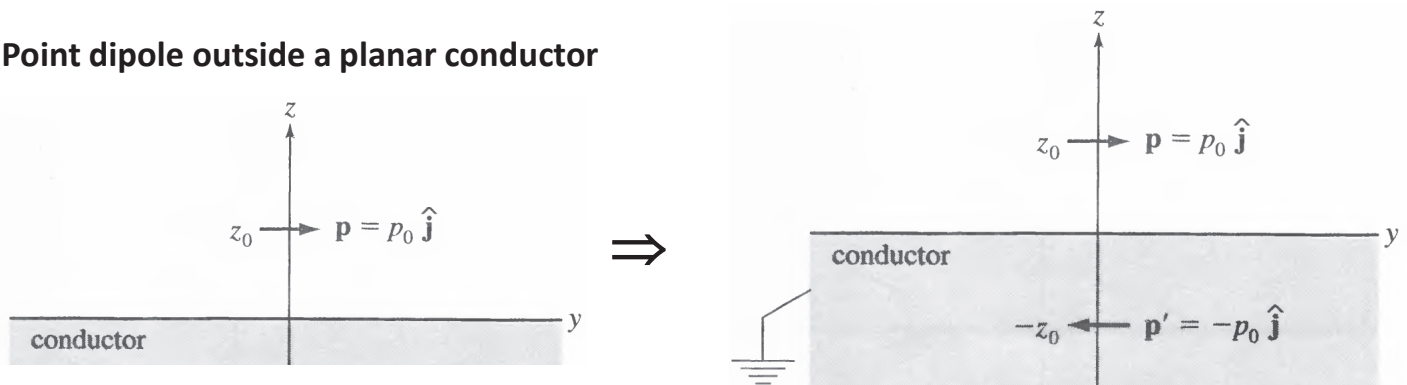
Since at a metal surface $\sigma = \varepsilon_0 E_n$ we then get: $\quad \sigma(r) = \dfrac{-qz_0}{2\pi(s^2 + z_0^2)^{3/2}}$

What is the force on the external charge? Since the response field of the conductor is the same as that of a charge $-q$ a distance $-z_0$ below the plane, the force on the external charge is just the force which would produce the image charge (!!) : $\quad \vec{F} = q\vec{E}_{response}(0,0,z_0) = -\dfrac{q^2}{16\pi\varepsilon_0}\dfrac{1}{z_0^2}\hat{k}$

Question for the class: how much is the force on the conductor?

Other examples of image charges

Point dipole outside a planar conductor

z

z_0 $\mathbf{p} = p_0\,\hat{\mathbf{j}}$

conductor

y

$\Rightarrow$

z

z_0 $\mathbf{p} = p_0\,\hat{\mathbf{j}}$

conductor

y

$-z_0$ $\mathbf{p'} = -p_0\,\hat{\mathbf{j}}$

Question 1: For a general orientation of the dipole, how does the image dipole look like? Is there a net force and a net torque on the dipole? In there an equilibrium orientation?
Question2: how are the image charges for a general charge distribution outside a metal plane?

Other example: point charge near an L-shaped conductor

z

$\bullet +q$

conductor

y

$\Rightarrow$

z

$-q \bullet$ $\bullet +q$

y

conductor

$+q \bullet$ $\bullet -q$
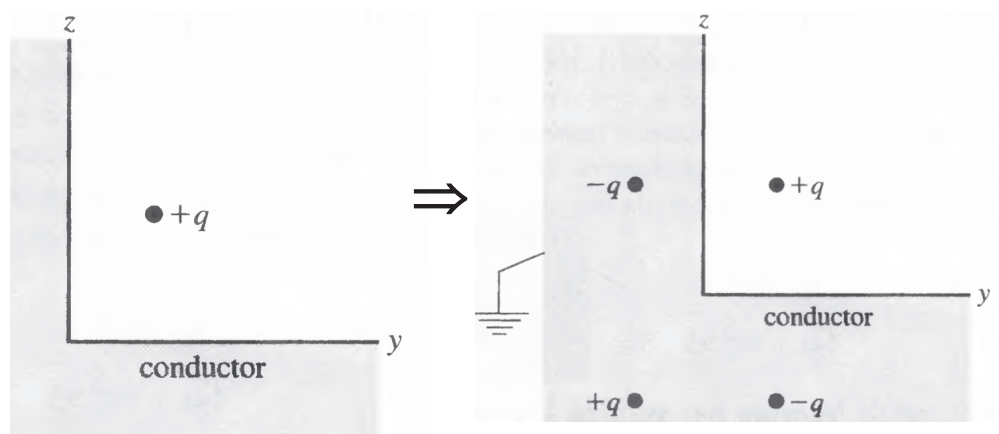
Conducting sphere in external E-field

Induced charge density on a neutral conducting sphere in uniform external field E_0 = E_0 z . The external potential due to E_0 is V_{ext} = $- E_0$ z. The zero of z is arbitrary; we take z = 0 in the sphere's center so that on the sphere's surface at a radius R_0 we have V_{ext} = $- E_0$ z = $- E_0 R_0 \cos\theta$. This is clearly not uniform on the conductor's surface; we need to find a set of image charges in the region occupied by the sphere such that the total potential V_{TOT} = V_{ext} + $V_{IMAGE\ CHARGE}$ is constant and = 0 on the sphere surface. It's easy if we remember that the potential of a (image) point dipole **p** centered in the origin is $V_{\vec{p}} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}\cdot\vec{r}}{r^3} = \dfrac{p\cos\theta}{4\pi\varepsilon_0 r^2}$ (p is parallel to z). The total potential is

$$V_{TOT}(r=R_0) = -E_0 R_0 \cos\theta + \frac{p\cos\theta}{4\pi\varepsilon_0 R_0^2}$$, which is zero if $p_{IMAGE} = 4\pi\varepsilon_0 R_0^3 E_0 \left(= 3\varepsilon_0 Vol_{sphere} E_0\right)$

The induced charge density is found from the boundary condition $\vec{E}_{TOT}\cdot\hat{n} = \vec{E}_{TOT}(r=R_0)\cdot\hat{r} = \dfrac{\sigma}{\varepsilon_0}$

Since the total field is the sum of the external field and the field of the image dipole, we get:

$$\sigma = \varepsilon_0\left(\vec{E}_0\cdot\hat{r} + \vec{E}_{\vec{p}}(r=R_0)\cdot\hat{r}\right) = \varepsilon_0 E_0\cos\theta + 2\varepsilon_0 E_0\cos\theta = 3\varepsilon_0 E_0\cos\theta = \sigma_0\cos\theta$$, that is:

$$\sigma(r=a) = \sigma_0\cos\theta \quad \text{with} \quad \sigma_0 = 3\varepsilon_0 E_0$$
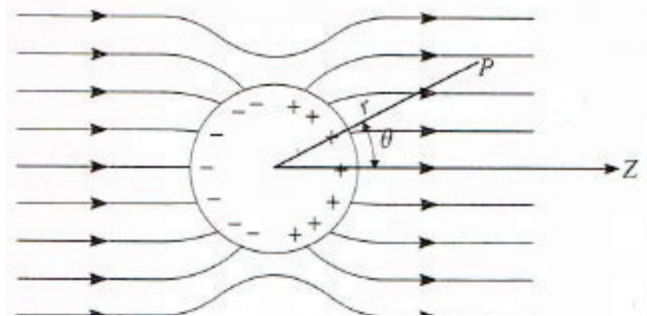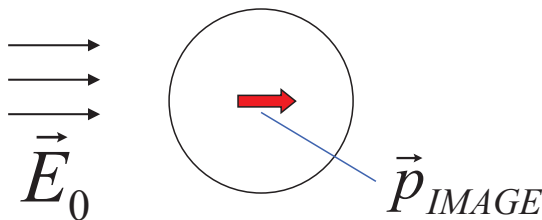
Question for the class: how much is the force on the sphere? How much is the torque?

Field of a spherical surface charge $\propto \cos\theta$

We saw in the previous slide that the induced charge density is $\sigma = \sigma_0\cos\theta$ with $\sigma_0 = 3\varepsilon_0 E_0$

Such a distribution must create a constant field equal to E_0 inside the sphere, so that E_{tot} inside:

$$\vec{E}_{TOT} = \vec{E}_0 + \vec{E}_{IMAGE} = 0 \Rightarrow \vec{E}_{IMAGE} = \vec{E}_\sigma = -\vec{E}_0 = -\frac{\sigma_0}{3\varepsilon_0}\hat{E}_0 = uniform$$

This result is very important and we will use it in several occasions (it is also on the formula list):

A spherical surface charge density proportional to cos(θ) produces a constant field inside the sphere and a dipolar field outside. If $\sigma = \sigma_0\cos\theta$, **then**

$$E_{inside} = \frac{\sigma_0}{3\varepsilon_0} \qquad E_{outside} = E_{\vec{p}} \text{ , with } p = 3\varepsilon_0 Vol_{sphere} E_{inside} = Vol_{sphere}\sigma_0$$
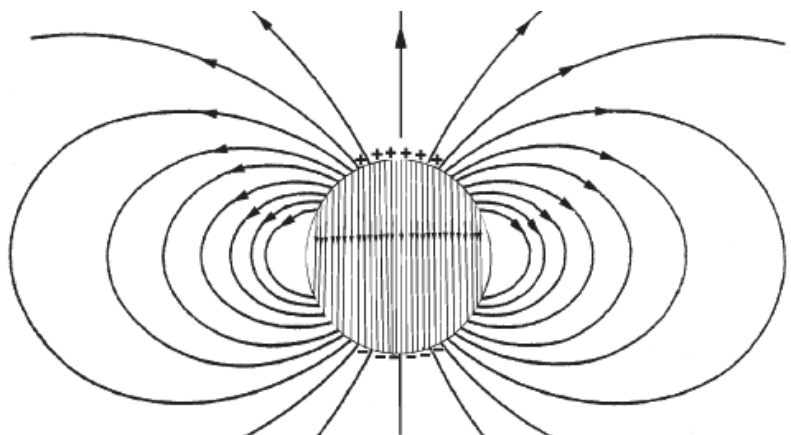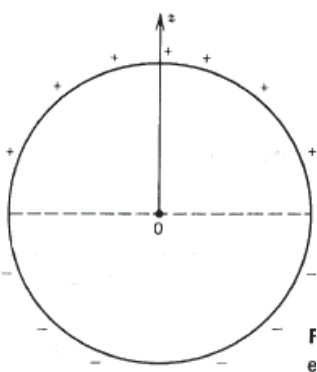
Exercise class: what happens if the sphere is at a potential V ≠ 0 or carries Q ≠ 0?

Conducting cylinder in external E-field

Induced charge density on an infinite neutral conducting cylinder (wire) of radius R_0 in uniform orthogonal external field E_0 (the cylinder axis is the z axis and the field is parallel to x). The external potential on the cylinder surface is $V_0 = -E_0 x$. We take $x = 0$ on the cylinder axis so that $V_0 = -E_0 x = -E_0 R_0 \cos\theta$. Note that now the angle θ is in cylindrical coordinates (and not spherical coordinates as before). The suitable image charge is now a *line* dipole $\wp$ extending on the cylinder's axis (parallel to z). The resulting total potential on the cylinder surface is

$$V_{TOT}(s = R_0) = -E_0 R_0 \cos\theta + \frac{\wp \cos\theta}{2\pi\varepsilon_0 R_0} \;,\; \text{which is zero if } \wp_{IMAGE} = 2\pi\varepsilon_0 R_0^2 E_0 \left(= 2\varepsilon_0 Area_{\substack{cross \\ section}} E_0 \right)$$

The induced charge density is found from the boundary condition $\vec{E}_{TOT} \cdot \hat{n} = \vec{E}_{TOT}(s = R_0) \cdot \hat{s} = \dfrac{\sigma}{\varepsilon_0}$
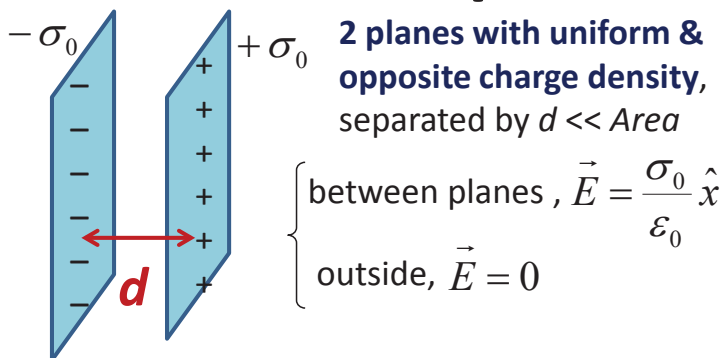
We get in this case: $\sigma = \varepsilon_0\left(\vec{E}_0 \cdot \hat{s} + \vec{E}_{\wp}(s = R_0) \cdot \hat{s}\right) = 2\varepsilon_0 E_0 \cos\theta = \sigma_0 \cos\theta$

Just as for the sphere case, this induced surface charge density must create a field equal and opposite to E_0 in the whole region occupied by the cylinder. Hence we find an important result for the field of a cylindrical surface charge proportional to $\cos\theta$ (in cylindrical coordinates):
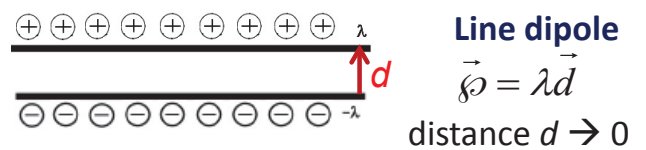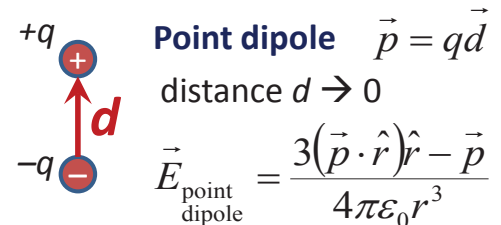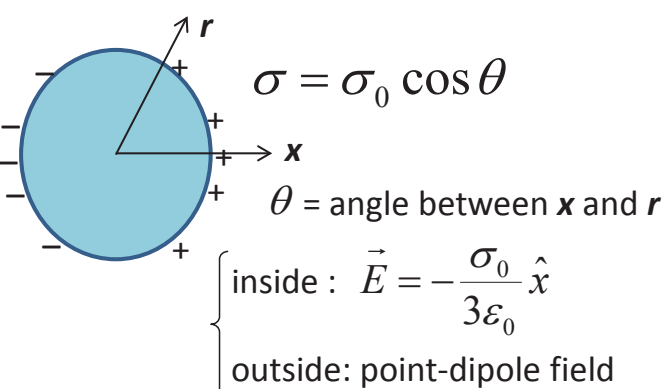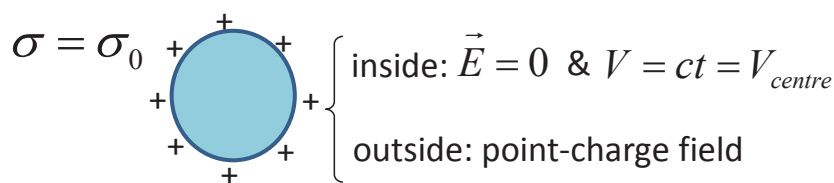
A cylindrical surface charge density proportional to cos(θ) generates a constant field inside the cylinder and a line-dipole field outside. If $\sigma = \sigma_0 \cos\theta$, then

$$\begin{cases} E_{inside} = \dfrac{\sigma_0}{2\varepsilon_0} \\[2mm] E_{outside} = E_{\vec{\wp}} \end{cases} \;,\; \text{with } \wp = 2\varepsilon_0 Area_{\substack{cross \\ section}} \quad E_{inside} = Area_{\substack{cross \\ section}} \sigma_0$$

E-field of special charge distributions

$-\sigma_0 \qquad +\sigma_0$

2 planes with uniform & opposite charge density, separated by $d \ll Area$

$$\begin{cases} \text{between planes,} \; \vec{E} = \dfrac{\sigma_0}{\varepsilon_0}\hat{x} \\[2mm] \text{outside,} \; \vec{E} = 0 \end{cases}$$

$+q$ $\qquad$ **Point dipole** $\quad \vec{p} = q\vec{d}$

$-q$ $\qquad$ distance $d \to 0$

$$\vec{E}_{\substack{point \\ dipole}} = \frac{3(\vec{p} \cdot \hat{r})\hat{r} - \vec{p}}{4\pi\varepsilon_0 r^3}$$

$\oplus \oplus \oplus \oplus \oplus \oplus \oplus \oplus \oplus$ λ

$\ominus \ominus \ominus \ominus \ominus \ominus \ominus \ominus \ominus$ $-\lambda$

Line dipole

$$\vec{\wp} = \lambda\vec{d}$$

distance $d \to 0$

$$\vec{E}_{\substack{line \\ dipole}} = \frac{2(\vec{\wp} \cdot \hat{s})\hat{s} - \vec{\wp}}{2\pi\varepsilon_0 s^2}$$

spherical surface charge densities:

$\sigma = \sigma_0$

$$\begin{cases} \text{inside:} \; \vec{E} = 0 \; \& \; V = ct = V_{centre} \\[2mm] \text{outside: point-charge field} \end{cases}$$

$\sigma = \sigma_0 \cos\theta$

θ = angle between x and r

$$\begin{cases} \text{inside :} \; \vec{E} = -\dfrac{\sigma_0}{3\varepsilon_0}\hat{x} \\[2mm] \text{outside: point-dipole field} \end{cases}$$

cylindrical surface charge density: $\sigma = \sigma_0 \cos\theta$

θ = angle between x and s

$$\begin{cases} \text{inside :} \; \vec{E} = -\dfrac{\sigma_0}{2\varepsilon_0}\hat{x} \\[2mm] \text{outside: line-dipole field} \end{cases}$$

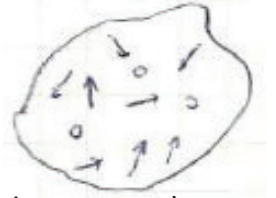# Dielectrics and polarized media (TOPIC 2)

When dealing with insulators (dielectrics), one usually wishes to calculate:
1) the field produced by a polarized dielectric
2) the effect of an applied field on a dielectric

Basic idea: ***a dielectric may be represented by a collection of point dipoles:***

A dielectric only carries a net charge if YOU charge it (by rubbing, ion bombardment, etc.)
Even approximating the atomic distributions as point dipoles, it's impossible to take into account all of them (there are of the order of $N_A \approx 10^{23}$ atoms/cm³ cube in a liquid/solid sample), let alone their mutual interactions. More than individual dipoles, we should consider the average dipole moment of a solid. We thus define the ***POLARIZATION*** of the material or medium as:

$$\vec{P} \equiv dipole \; moment \; per \; unit \; volume = \frac{1}{\delta\tau}\sum_{i=1}^{\delta N}\vec{p}_i \qquad \textcolor{red}{\textbf{\textit{Polarization field}}}$$

Similarly, instead of calculating the local electric field on atomic scale, we should calculate an average, ***macroscopic*** electric field due to all the dipoles. This can be done with the concept of bound charge. Take for example a rectangle of atoms that have dipole moment parallel to two of the rectangle's sides:
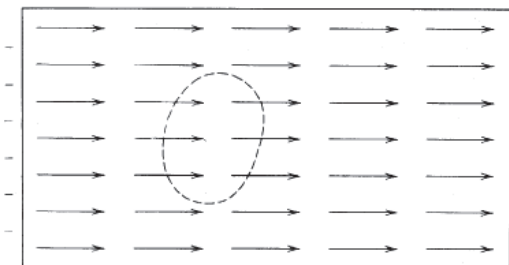
As you can see in the figure, the end of a dipole (+ charge) is close to the start of another dipole (− charge) at all points inside the rectangle, so that the average field of due to both charges is zero. This is not true at the surface, where there are "uncompensated" charges. These charges, called ***BOUND CHARGES***, produce the E-field inside and outside the sample

Fundamental theorem of polarized media

The procedure suggested in the previous slide is possible in general. As mentioned, a polarized material may be considered as a collection of point dipoles, and whose volume density is the polarization field:
$$\vec{P} = \frac{1}{\delta\tau}\sum_{i=1}^{\delta N}\vec{p}_i \Rightarrow \vec{P}(\vec{r}) = \frac{d\vec{p}(\vec{r})}{d\tau} \qquad \text{SI units of } \textbf{P} : \text{C/m}^2$$

Each elementary volume $d\tau$ of the material has thus an elementary dipole moment $d\vec{p} = \vec{P}d\tau$
What is the E-field produced by a polarized object? For a point dipole we have $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$
Hence for a continuous collection of dipoles each at position $\vec{r}$:

$$V(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau \frac{\vec{P}(\vec{r})\cdot(\vec{r}'-\vec{r})}{|\vec{r}'-\vec{r}|^3} = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau \, \vec{P}(\vec{r})\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}'-\vec{r}|}\right) \qquad \text{(since } \vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}\text{)}$$

Integrating by parts $\vec{\nabla}(f\,\vec{G}) = f\cdot\vec{\nabla}G + \vec{G}\cdot\vec{\nabla}f$ and using the divergence theorem:

$$V(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\int_{volume}\left[\vec{\nabla}\cdot\left(\frac{\vec{P}}{|\vec{r}'-\vec{r}|}\right) - \frac{1}{|\vec{r}'-\vec{r}|}(\vec{\nabla}\cdot\vec{P})\right]d\tau = \frac{1}{4\pi\varepsilon_0}\oint_{surface}\frac{\vec{P}}{|\vec{r}'-\vec{r}|}\cdot\hat{n}da - \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{1}{|\vec{r}'-\vec{r}|}(\vec{\nabla}\cdot\vec{P})d\tau$$

The last equality may be written as: $V(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\oint_S\frac{\sigma_b}{|\vec{r}'-\vec{r}|}da + \frac{1}{4\pi\varepsilon_0}\int_V\frac{\rho_b}{|\vec{r}'-\vec{r}|}d\tau$

Here we have defined $\sigma_b = \vec{P}\cdot\hat{n}$ and $\rho_b = -\vec{\nabla}\cdot\vec{P}$ (surface/volume "bound charge density")

the electrostatic potential and field produced by a polarized medium of polarization P is the same as that produced by a volume charge density $\rho_b = -\vec{\nabla}\cdot\vec{P}$ plus a surface charge density $\sigma_b = \vec{P}\cdot\hat{n}$

Physical meaning of bound charges

What is this "bound charge", and what is the difference with the charge in a conductor? In an atom one distinguishes between core electrons and valence electrons (those in the outer shell); when atoms form a solid, they share their valence electrons: in some solids ("metals") these electrons can move freely from one atom to the next; in others ("dielectrics" or "insulators") valence electrons are tightly bound to a single atom or bind together two atoms. In the first case, the valence electrons are *free charges*, in the sense that they can move anywhere; in the second, they are *bound charges*, in the sense that they cannot be separated from the atom or pair of atoms they are linked to. Bound charges can only move slightly with respect to the nuclear charge they are bound to in an applied field; this is what we call "*induced polarization*"
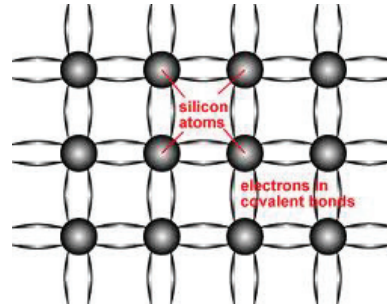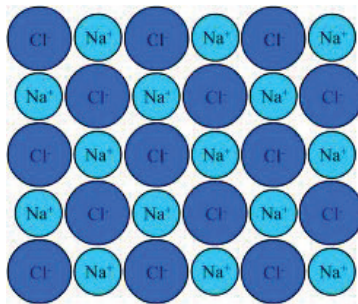
Notice that even in an insulator there can be free charges, for example if it has been electrized: in fact, an excess or lack of one electron on a site is a free charge that can move about, if all other sites are neutral (think about what happens in a semiconductor). If there are no free charges, but only bound charges inside dipoles, then the material carries no net charge, that is, it is neutral:

$\vec{P}$ is a macroscopic field

The polarization field $\vec{P}$ is an average, "macroscopic" field. We may see it by rewriting it as:

$$\vec{P} = \frac{\sum_i^{\delta N} \vec{p}_i}{\delta\tau} = \frac{\delta N}{\delta\tau}\frac{\sum_i^{\delta N} \vec{p}_i}{\delta N} = \frac{\delta N}{\delta\tau}\langle\vec{p}\rangle = n\langle\vec{p}\rangle$$

(The SI units of **P** are the same as those of a surface charge density, namely [P] = C m^{-2})

We see that $\vec{P}$ is equal to the number density of atoms $n = \dfrac{\delta N}{\delta\tau}$ times the average electric dipole moment per atom, given by $\langle\vec{p}\rangle = \dfrac{\sum_i^{\delta N} \vec{p}_i}{\delta N}$. Since $\rho_b = -\vec{\nabla}\cdot\vec{P}$ and $\sigma_b = \vec{P}\cdot\hat{n}$, we

see that **the bound charge densities ρ_b and σ_b are average, macroscopic charge densities**
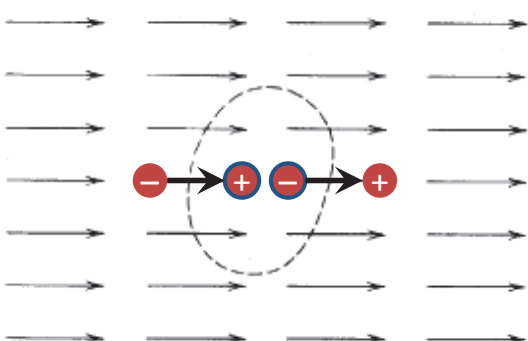
It is precisely because we are considering averages that we can say, for example in the case of uniform polarization, that the bound charge at the end of a dipole (+ charge) cancels out exactly the bound charge at the start of the dipole next to it (− charge).

"physical" proof of the fundamental theorem

Consider a small area *da* at the boundary or inside a dielectric. When the dielectric becomes polarized, positive and negative charges are separated by $\vec{\ell}$ and some charge dQ_{across} will cross *da*. The atoms that cross *da* are those that lie at a distance $\ell_n/2$ of *da*, where $\ell_n = \hat{n} \cdot \vec{\ell}$ is the component of $\vec{\ell}$ normal to *da*. dQ_{across} is the number of atoms in the volume $d\tau = \ell_n da$, multiplied by the charge *q*, that is:

$$dQ_{across} = qn_a\ell_n da \qquad (n_a \text{ is the atomic density})$$
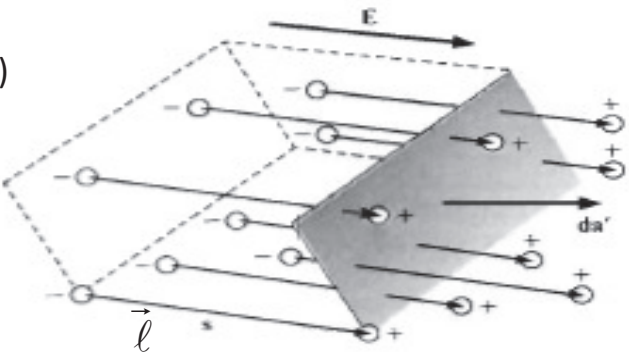
Since $qn_a\ell_n$ is the normal component of the polarization field, we get:

$$dQ_{across} = P_n da$$

If *da* is a part of the surface of the dielectric, dQ_{across} will correspond to a surface bound charge.

The *surface* bound charge density is $\dfrac{dQ_{across}}{da} = P_n = \hat{n} \cdot \vec{P} = \sigma_b$ (this proves part of the theorem)

If instead *da* is part of the closed frontier S of a volume dV, then $Q_{across} = \oint_S P_n da = \int_V \vec{\nabla} \cdot \vec{P} d\tau$

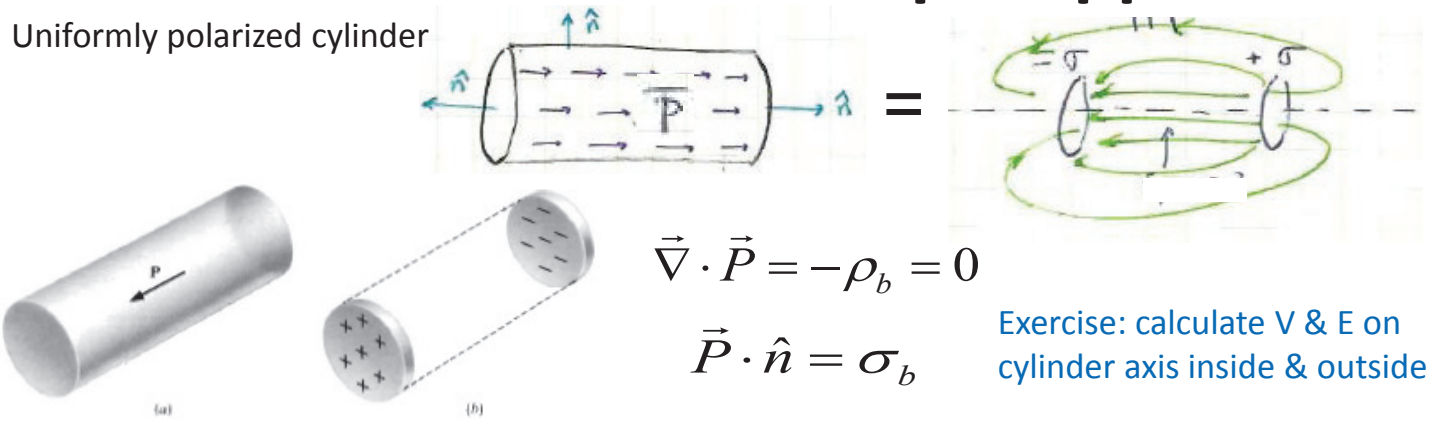Since charge is conserved, the charge crossing S is the charge leaving V:

$$Q_{across} = -Q_{inside} = -\int_V \rho_b d\tau \qquad \text{where } \rho_b \text{ is the } volume \text{ bound charge density}$$

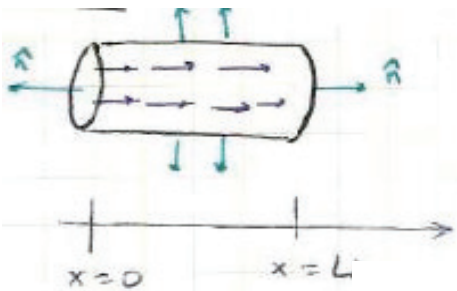Comparing the last two equations for Q_{across} we see that $\rho_b = -\vec{\nabla} \cdot \vec{P}$

Fundamental theorem:simple applications

Uniformly polarized cylinder

$$\vec{\nabla} \cdot \vec{P} = -\rho_b = 0$$
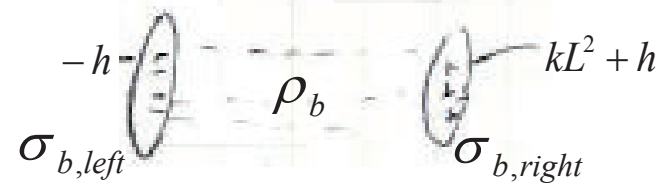
$$\vec{P} \cdot \hat{n} = \sigma_b$$

Exercise: calculate V & E on cylinder axis inside & outside

Non-uniformly polarized cylinder of length L with $\vec{P}(x) = (kx^2 + h)\hat{x}$

bound charge volume density: $\quad \vec{\nabla} \cdot \vec{P}(\vec{r}) = \dfrac{dP(x)}{dx} = 2kx = -\rho_b(x)$
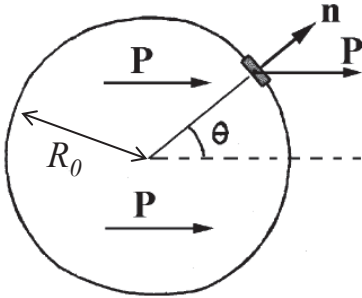
bound charge surface density: $\quad \begin{array}{cc} \vec{P} \cdot \hat{n} = -h & \vec{P} \cdot \hat{n} = kL^2 + h \\ (\text{left surface, } \hat{n} = -\hat{x}) & (\text{right surface, } \hat{n} = \hat{x}) \end{array}$

Exercise: verify that $Q_b = 0$
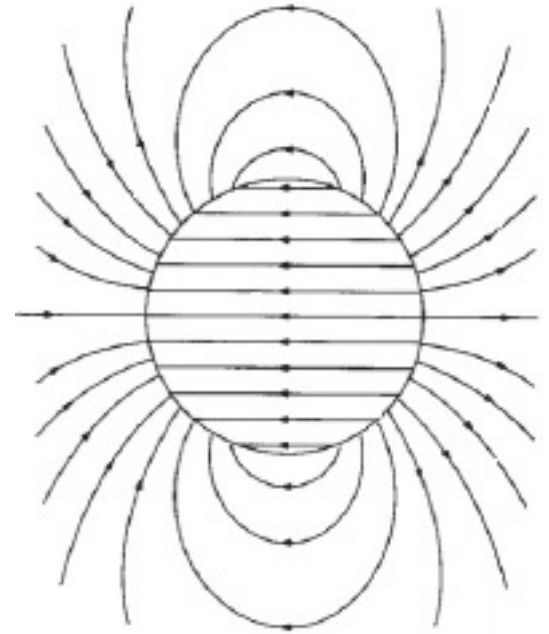
Uniformly polarized sphere of radius R_0

$$\vec{\nabla} \cdot \vec{P} = -\rho_b = 0$$

$$\vec{P} \cdot \hat{n} = P \cos \theta = \sigma_b$$

We already encountered this surface distribution when solving the problem of a conducting sphere in an applied field. We know (see also the formula list) that generates a uniform field opposite to P inside the sphere and a dipolar field outside: →

$$\sigma = \sigma_0 \cos \theta$$
$$(= P \cos \theta)$$
$$\Longrightarrow \begin{cases} \vec{E}_{inside} = -\dfrac{\sigma_0}{3\varepsilon_0} \hat{x} \left(= -\dfrac{\vec{P}}{3\varepsilon_0} \right) \\ \vec{E}_{outside} = \vec{E}_{\vec{p}} \quad \text{with :} \\ \vec{p} = \vec{P} \cdot Vol_{sphere} = \dfrac{4}{3} \pi R_0^3 \vec{P} \end{cases}$$

there exist two types of dielectrics:
→ Ferroelectrics (spontaneous intrinsic polarization)
→ Linear dielectrics (induced polarization)

Linear dielectrics

A linear dielectric is one in which the polarization is directly proportional to the <u>macroscopic</u> (average total) electric field inside the sample: **P = ε_0 χ E$_{macro}$**
The constant of proportionality is ε_0 times a pure number, χ, called ELECTRIC SUSCEPTIBILITY
This equation is only apparently simple, because the macroscopic field is itself an unknown, and depends on the polarization P! To see it, let's consider a typical problem: a piece of linear dielectric material in a uniform applied external field E$_{ext}$. We want to find P as function of E$_{ext}$.
It is tempting but **WRONG** to write P = $\varepsilon_0 \chi$ E$_{ext}$, since the macroscopic field is NOT equal to E$_{ext}$. We need to be more careful.
Example: dielectric in the shape of a very thin disk or slab
Let's <u>assume</u> that the induced polarization is parallel to the applied field and constant (we try and see if it works). The polarization creates a field (called *depolarizing field*) which may be calculated from the bound charges (two (quasi-infinite) sheets with uniform and opposite surface charge density
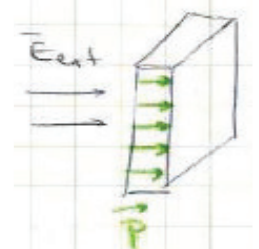
$$\vec{\nabla} \cdot \vec{P} = -\rho_b = 0$$
$$\vec{P} \cdot \hat{n} = \sigma_b = \pm |P|$$

In the present case we find: $E_{depolarizing} = \dfrac{\sigma}{\varepsilon_0} = \dfrac{|P|}{\varepsilon_0}$
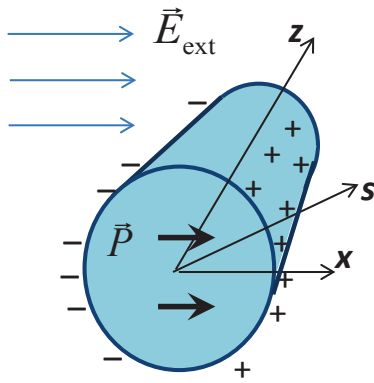(only true for a uniformly polarized infinite slab)
Hence from the definition of linear medium we get:

$$\vec{P} = \chi\varepsilon_0 \vec{E}_{TOT} = \chi\varepsilon_0 \left(\vec{E}_{ext} + \vec{E}_{depolarizing}\right) = \chi\varepsilon_0 \left(\vec{E}_{ext} - \dfrac{\vec{P}}{\varepsilon_0}\right) = \vec{P} \quad \Rightarrow \quad \vec{P} = \dfrac{\chi\varepsilon_0}{1+\chi} \vec{E}_{ext}$$

The quantity $\varepsilon_r = 1 + \chi$ is called **relative permittivity**. We thus get: $\vec{P} = \left[(\varepsilon_r - 1)\varepsilon_0 / \varepsilon_r\right] \vec{E}_{ext}$

The proportionality factors we found are NOT general, they depend on the dielectrics' shape →

Cylinder of linear dielectric material in E_{ext}

Let's assume that the induced polarization **P** is uniform. Then there is no bound volume charge, but only a bound surface charge: $\sigma_b = \vec{P} \cdot \hat{n} = |P| \cos \theta$. We therefore get:

$$\vec{P} = \chi \varepsilon_0 \left(\vec{E}_{ext} + \vec{E}_{depolarizing} \right) = \chi \varepsilon_0 \left(\vec{E}_{ext} - \frac{\vec{P}}{2\varepsilon_0} \right) \Rightarrow$$

$$\vec{P}\left(1 + \frac{\chi}{2}\right) = \chi \varepsilon_0 \vec{E}_{ext} \Rightarrow \vec{P}_{induced} = \frac{\chi \varepsilon_0}{1 + \frac{\chi}{2}} \vec{E}_{ext} = 2 \frac{\varepsilon_r - 1}{\varepsilon_r + 1} \varepsilon_0 \vec{E}_{ext}$$

IMPORTANT: the trick of assuming that he induced polarization is uniform does not always work In fact it almost never works! For example take the same cylinder under an applied <u>axial</u> E-field:
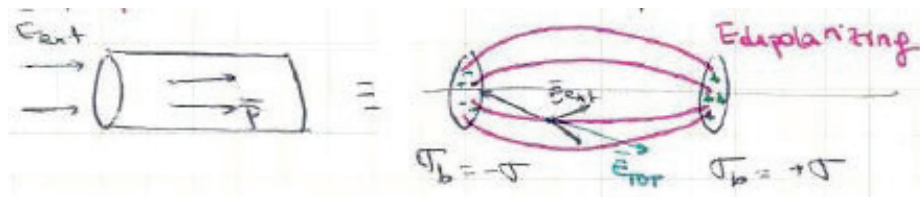
If we assume that P is uniform, then the depolarizing field would be that created by two disks, which is not uniform.

But then the relationship $\vec{P} = \chi \varepsilon_0 \vec{E}_{tot} = \chi \varepsilon_0 (\vec{E}_{ext} + \vec{E}_{depolarizing})$ cannot hold, since P and E_{ext} are uniform, but $E_{depolarizing}$ isn't! (Another way of seeing this is calculating graphically the total field $E_{ext} + E_{depolarizing}$ and noticing that it is not parallel to P). Hence our initial assumption that P is uniform is wrong !! The problem of finding P for an arbitrary shape is complicated.

Exercise: what happens in this last case if the cylinder is very long & thin (needle-like)?

Displacement field (D)

Many problems with dielectrics involve also <u>free charges</u>, both outside (external charges) and inside the dielectric. (An external field acting on a dielectric may be caused by a set of free external charges in vacuum or on conductors nearby the dielectric. On the other hand, a dielectric may contain free charges if it has been *electrized* , for example by rubbing it or bombarding it with ions or with electrons.) In a typical problem with dielectrics and conductors, for example, the polarization is initially unknown; however also the distribution of free charge on the conductors is then not known in general, since it will be affected by the polarized dielectric. This complicated problem may be simplified by introducind an auxiliary field D, defined as follows. First, consider that in such a problem the total macroscopic field E is generated by both free and bound charges. Hence:

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f}{\varepsilon_0} + \frac{\rho_b}{\varepsilon_0}$. Using $\rho_b = -\vec{\nabla} \cdot \vec{P}$, we can write Gauss's law as $\vec{\nabla} \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f$

We define the (macroscopic) D field as: $\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$ SI units of **D** = units of **P** = C/m²

It follows that the field equations for D are: $\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f}$ & $\boxed{\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}}$

{NOTE: "extra" charges inside a dielectric are free charges, since although they move slowly, they are not bound to an equal and opposite charge: if an atom of the dielectric is ionized by adding an electron, this extra electron can jump to any neighboring atom and thus move through the material without an associated positive charge moving with it (it is then a free charge). The same happens when the ionization occurs by loss of one electron: if an electron from a nearby atom fills the vacancy, it leaves another vacancy behind, hence in such case there is a positive free charge (called vacancy or **hole**) moving freely in the material.}

Linear dielectrics and dielectric constant

In a **linear** dielectric, the polarization is directly proportional to the macroscopic field E inside the medium: $\vec{P} = \chi_{el}\varepsilon_0\vec{E} \implies \vec{D} = \varepsilon_0\vec{E} + \chi\varepsilon_0\vec{E} = \underbrace{(1+\chi)}_{\varepsilon_r}\varepsilon_0\vec{E} = \varepsilon_r\varepsilon_0\vec{E}$

The equation linking **D** and **E** is called **constitutive relation**

$\varepsilon_r = 1 + \chi_{el}$: **relative dielectric constant** or **relative permittivity**

Material	Dielectric constant ε_r	Dielectric strength E_{max} in 10^6 V/m
air	1.00059	3
polystyrene	2.5	20
Lucite	2.8	20
Plexiglas	3.4	40
Teflon	2.1	60
Mylar	3.1	
paper	3.7	16
fused quartz	3.8 to 4.1	
Pyrex	4 to 6	14
water	80	
strontium titanate	332	8

$\varepsilon_r \rightarrow$ dimensionless number

material	ε_r
Benzene	2.28
Diamond	5.7
Salt	5.9
Silicon	11.8
Methanol	33.0
Water	80.1
Ice (-30° C)	99
KTaNbO$_3$ (0° C)	34,000

IMPORTANT: in a <u>perfect</u> metal there are no polarization effects, hence $\vec{P} = 0$ and $\vec{D} = \varepsilon_0\vec{E}$, just as in vacuum. In other words, ε_r **= 1 in a metal**

Example: a linear dielectric in a capacitor

Linear homogeneous dielectric inside a parallel plate capacitor held at a potential difference V
By symmetry, the free charge density on each plate is uniform. Hence it is straightforward to determine D from Gauss's law for **D** :

$\sigma_f = -\sigma$

$\vec{\nabla} \cdot \vec{D} = \rho_f$
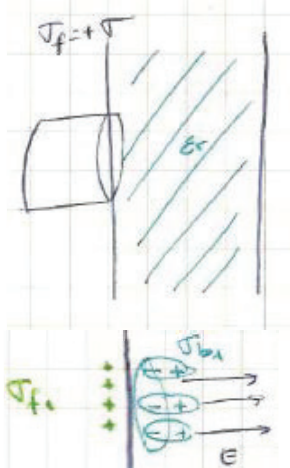
$\rightarrow \vec{D} = \sigma\,\hat{x} = |\sigma_f|\,\hat{x}$

$E = \dfrac{D}{\varepsilon_r\varepsilon_0} = \dfrac{\sigma_f}{\varepsilon_r\varepsilon_0}$

The boundary condition for E yields $\sigma_{tot} = \sigma_f/\varepsilon_r$

For *V* we have $V = \int_-^+ \vec{E}\cdot d\vec{\ell} = Ed$

Hence $V = \sigma_f\,d/\varepsilon_r\,\varepsilon_0$. If the capacitor has total area A, then $\sigma_f = Q_f\,/A$, where Q_f is the total free charge on one plate. Hence:
$V = Q_f\,d/(A\,\varepsilon_r\,\varepsilon_0)$, or: $C = Q_f\,/V = \varepsilon_r\,\varepsilon_0\,A/d$

From these results we can calculate also everything about the dielectric, namely P and σ_b

The capacitance increases from its value $C_0 = \varepsilon_0\,A/d$ when the capacitor is empty, to the value $C = \varepsilon_r\,C_0$ when the capacitor is filled with dielectric. Why is the capacitance higher?
The physical cause of this is the depolarization field, or if you prefer, the bound charge density at the surface of the dielectric, which effectively reduces the magnitude of the total E inside the dielectric, so that more free charge is needed to obtain the same voltage across the capacitor.
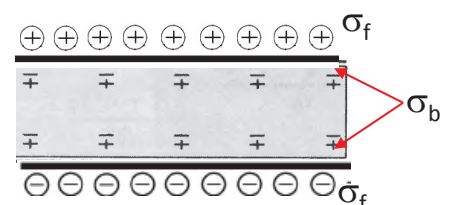
Question for the class: why can we neglect the other eq. for D? :
$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$$

Boundary conditions&charge in dielectrics

$$\vec{\nabla} \times \vec{E} = 0 \implies E_{t1} = E_{t2}$$

$$\vec{\nabla} \cdot \vec{D} = \rho_f \implies \vec{D}_1 \cdot \hat{n}_1 + \vec{D}_2 \cdot \hat{n}_2 = \sigma_f$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \implies \vec{E}_1 \cdot \hat{n}_1 + \vec{E}_2 \cdot \hat{n}_2 = \sigma / \varepsilon_0$$

or $D_{2n} - D_{1n} = \sigma_f$

We'll use the same tool (boundary conditions) also in other topics, for other interfaces & fields

$$\left. \begin{array}{l} \vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \\ \vec{\nabla} \cdot \vec{D} = \rho_f \end{array} \right\} \implies \rho_f = \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\varepsilon_r \varepsilon_0 \vec{E}) = \varepsilon_r \varepsilon_0 \vec{\nabla} \cdot \vec{E} = \varepsilon_r \varepsilon_0 \frac{\rho_{tot}}{\varepsilon_0} = \varepsilon_r \rho_{tot}$$

$$\implies \rho_f = \varepsilon_r \rho_{tot} \quad \longleftarrow \quad \text{for a homogeneous medium}$$

Poisson's law in a homogeneous medium then becomes $-\vec{\nabla}^2 V = \dfrac{\rho_{tot}}{\varepsilon_0} = \dfrac{\rho_f}{\varepsilon_r \varepsilon_0}$

Hence **if a homogeneous dielectric is not charged (ρ_f= 0), Laplace's law $\vec{\nabla}^2 V = 0$ holds**

Simple relation (only valid at the interface of a conductor with a linear dielectric) → since inside D = E = 0, and outside both fields are normal to the surface, we have:

$$E_{\text{out}} = \sigma / \varepsilon_0 \quad \& \quad D_{\text{out}} = \sigma_f$$

$$\vec{D} = \varepsilon_r \varepsilon_0 \vec{E} \implies \sigma_f = \varepsilon_r \sigma_{tot} \quad \longleftarrow \quad \text{only at a conductor's surface!!}$$

Problems with 2 dielectrics in capacitors

2 types of problems:
(a) highly symmetric, with the separation between media running orthogonal to E
(b) Less symmetric, with the separation between media running parallel to E

Strategy for case (a): guess the symmetry of D and σ_f ; calculate the relation between D and Q_f, then apply constitutive relation to find E; integrate to get *V* as function of Q_f (whence $C = Q_f/V$)

Strategy for case (b): study the boundary conditions for E to guess the symmetry of E (which will be the same as that of the empty capacitor); get *V* and the total surface charge density σ; hence apply constitutive relation to find D. and from there get σ_f and Q_f to obtain the expression of *C*
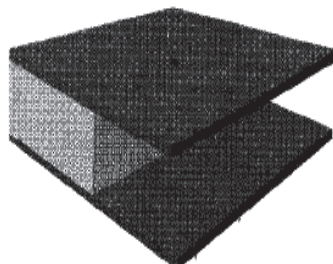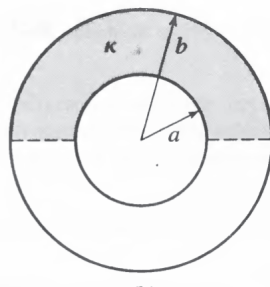

IMPORTANT: in the definition of the capacitance *C*, Q is the total charge on the metal, that is, the free charge :

$$\boxed{C = \frac{Q_f}{V}}$$
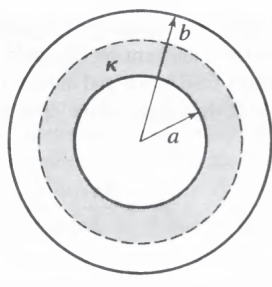
Electrostatic energy with linear dielectrics

The electrostatic energy for a set of conductors and LINEAR dielectrics is:

$$U = \frac{1}{2} \int_{volume} d\tau \, \rho_f(\vec{r}) \, V(\vec{r}) + \frac{1}{2} \int_{surfaces} dA \, \sigma_f(\vec{r}) \, V(\vec{r})$$

Here $V(\vec{r}) = V_{q_f,q_b}(\vec{r})$ is the **total potential due to both free and bound charges**.

The first integral is over the volume that is NOT occupied by the conductors, since $\rho_f = 0$ inside a metal. The second integral is over the surfaces of the conductors. Using $\vec{\nabla} \cdot \vec{D} = \rho_f$ in the 1st integral and $\vec{D} \cdot \hat{n} = \sigma_f$ in the 2nd (which is valid at a metal surface since **D** = 0 inside) we get, integrating by parts:

$$U = \left[\frac{1}{2} \int d\tau \, V(\vec{\nabla} \cdot \vec{D}) \right] + \frac{1}{2} \int dA (V\vec{D}) \cdot \hat{n}$$

$$= \vec{\nabla} \cdot (V\vec{D}) - \vec{D} \cdot (\vec{\nabla} V) = \vec{\nabla} \cdot (V\vec{D}) + \vec{D} \cdot \vec{E}$$

The first term in U is the volume integral of a divergence, which due to Gauss' integral theorem is equal to the flux of the product (V **D**) through the surface delimiting the volume in between conductors. But this flux is exactly equal to the last term in U changed in sign (since the direction of the normal vector is inward the metals in the first case, outward the metal in the second case). Hence we get:

$$U = \frac{1}{2} \int d\tau \vec{D} \cdot \vec{E}$$

We can write our result in terms of the energy density u_{el}, defined as: $U = \int d\tau \, u_{el}$

We find:

$$\boxed{u_{el} = \frac{1}{2} \rho_f V = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1}{2} \varepsilon_r \varepsilon_0 E^2}$$

Force on metal surfaces and on dielectrics

Since electrostatic forces are conservative, one way to calculate them is as gradients of the electrostatic energy. This is particularly useful in the case of a capacitor, whose electrostatic energy is simply: $U_{el} = \frac{1}{2} Q_f V = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q_f^2}{C}$. Consider for example a parallel plate capacitor.

The plates carry opposite charge and are therefore attracted to one another. We can calculate the attractive force on one of the plates using the following method: take one of the plates to be fixed and the other one to be mobile, and take the total charge on each one to be constant (that is, the plates are electrically isolated). Because of energy conservation, the work done by electrostatic forces to pull the mobile plate by a distance dx must equal minus the variation of electrostatic energy of the configuration. In other words, the force is simply: $F_{el} = -\frac{d}{dx} U_{el}$. Since the charge is constant, this can be written as:

$$F_{el} = -\frac{d}{dx}\left(\frac{1}{2} \frac{Q_f^2}{C} \right) = -\frac{1}{2} Q_f^2 \frac{d}{dx}\left(\frac{1}{C} \right) = \frac{1}{2} \frac{Q_f^2}{C^2} \frac{dC}{dx} = \frac{1}{2} V^2 \frac{dC}{dx}$$

Although we calculated this force assuming that we were holding one of the plates fixed and maintaining the free charged fixed as well, the electrostatic force is determined entirely by the distribution of charge, free and bound. In other words, it cannot possibly depend on how you plan to measure it, whether by keeping one plate fixed or not, or whether they are connected to a power supply instead of being isolated. Therefore, the formula that we found is actually always true. Moreover, if instead of considering a moving plate we consider a dielectric that only partially fills the volume of a capacitor, the same formula applies, and in such case it gives the force <u>on the dielectric</u> ! (see next slide)

Electrostatic force and pressure

We saw in the previous slide that the force in a capacitor is $\boxed{F_{el} = -\frac{1}{2}Q_f^2 \frac{d}{dx}\left(\frac{1}{C}\right) = \frac{1}{2}V^2 \frac{dC}{dx}}$

For example, for a parallel plate capacitor completely filled with a dielectric of relative permittivity ε_r, if the plates are separated by a distance x we have we have $C = \varepsilon_r \varepsilon_0 \frac{Area}{x}$

Hence we get:
$$F_{el} = -\frac{1}{2}Q_f^2 \frac{d}{dx}\left(\frac{1}{C}\right) = \frac{1}{2}Q_f^2 \frac{d}{dx}\left(\frac{x}{\varepsilon_r \varepsilon_0 Area}\right) = \frac{1}{2}\frac{Q_f^2}{\varepsilon_r \varepsilon_0 Area} = \frac{1}{2\varepsilon_r \varepsilon_0}\sigma_f^2 Area$$

This results suggests defining an "electrostatic pressure" on the conducting plates as

$$p_{el} = \frac{F_{el}}{Area} = \frac{1}{2\varepsilon_r \varepsilon_0}\sigma_f^2 = \frac{1}{2\varepsilon_r \varepsilon_0}D^2 = \frac{1}{2}\vec{D}\cdot\vec{E}$$

the electrostatic pressure is thus equal to the electrostatic energy density, $p_{el} = u_{el}$!
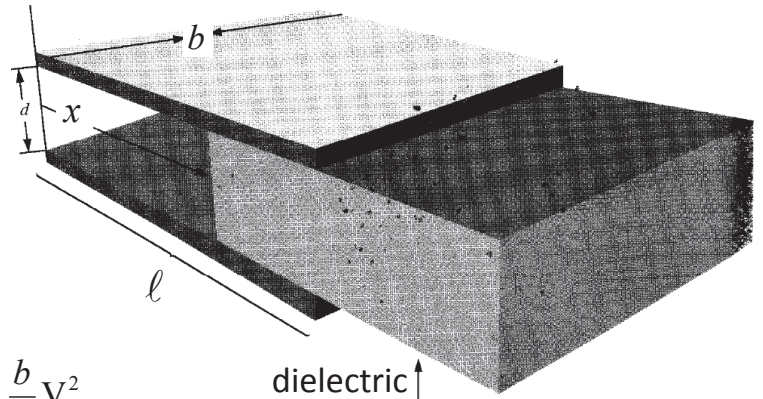
Other example: a dielectric is partially inserted in a parallel-plate capacitor. In terms of x (length of empty portion of the capacitor), the capacitance is:

$$C = \varepsilon_0 \frac{bx}{d} + \varepsilon_r \varepsilon_0 \frac{b(\ell-x)}{d} = \varepsilon_r \varepsilon_0 \frac{b}{d}\ell + \varepsilon_0 \frac{b}{d}x(1-\varepsilon_r)$$

Hence:

$$F_{el} = \frac{1}{2}V^2 \frac{dC}{dx} = \frac{1}{2}(1-\varepsilon_r)\varepsilon_0 \frac{b}{d}V^2 = -\frac{1}{2}(\varepsilon_r - 1)\varepsilon_0 \frac{b}{d}V^2$$

The fact that F < 0 means that the force tends to pull in the dielectric

dielectric ↑

Applications:
-electrostatic levitation
-polymer patterning

Microscopic theory of ε_r for linear media

The field E_{local} felt by an atom of a dielectric is the sum of the external field E_{ext} and the field of all other dipoles. In the simplest case, the field E_{local} felt by each atom is the same, so that the induced microscopic dipole moments are all equal and given by $\mathbf{p} = \alpha \mathbf{E}_{local}$. The macroscopic polarization is then $\vec{P}(\vec{r}) = n\vec{p}$. Let's calculate the relationship between ε_r and α:

Case 1: dilute apolar gas In a gas the atoms or molecules are relatively far from one another, so that we can consider that the field they produce does not affect other atoms. Hence $E_{local} = E_{ext}$

Then, with $\vec{E} = \vec{E}_{ext}$: $\vec{P} = n\vec{p} = n\alpha\vec{E} = \chi\varepsilon_0\vec{E} \Rightarrow n\alpha = \chi\varepsilon_0 \Rightarrow \boxed{\alpha = \frac{\varepsilon_0}{n}(\varepsilon_r - 1) \Rightarrow \varepsilon_r = 1 + \frac{n\alpha}{\varepsilon_0}}$

Case 2: dense apolar liquid/solid

In condensed matter E_{local} differs considerably from the applied external field and also from the macroscopic (average) field E_{macro} inside the dielectric. We use the following approximation: we take E_{local} to be equal to the macroscopic field inside the dielectric plus the field produced by the walls of a small spherical cavity carved inside the polarized dielectric, which is assumed to have a locally uniform polarization P. The field due to the cavity walls is generated by the bound charge density, which goes like $\cos\theta$; hence such field is equal to $+P/3\varepsilon_0$ (and is parallel to $+P$).
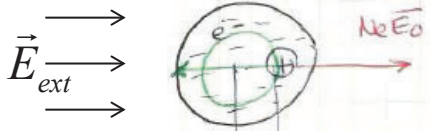
Therefore: $\vec{P} = n\vec{p} = n\alpha\vec{E}_{local} = n\alpha\left(\vec{E}_{macro} + \frac{\vec{P}}{3\varepsilon_0}\right) \Longrightarrow \vec{P}\left(1 - \frac{n\alpha}{3\varepsilon_0}\right) = n\alpha\vec{E} \Rightarrow \vec{P} = \left(\underbrace{\frac{n\alpha}{1-n\alpha/3\varepsilon_0}}_{=\chi_{el}\varepsilon_0}\right)\vec{E}$

$\Rightarrow \chi_{el} = \frac{n\alpha}{\varepsilon_0 - n\alpha/3}$ & $\boxed{\alpha = \frac{\varepsilon_0}{n}\frac{\chi_{el}}{1+\chi_{el}/3} = \frac{3\varepsilon_0}{n}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}}$ **Clausius-Mossotti relation**
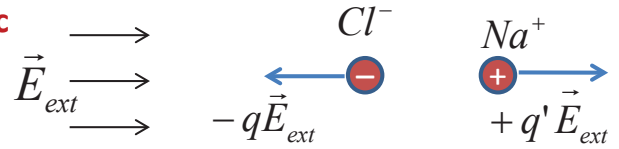
The Clausius-Mossotti relation holds for liquids, glasses, and cubic crystals. It reduces to case (1) for $\varepsilon_r \approx 1$
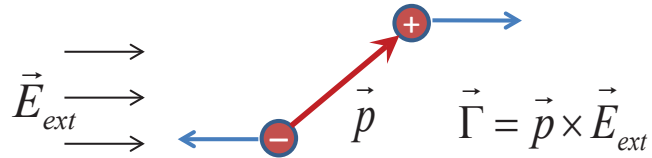
Mechanisms of induced polarization

(1) Electronic

$\vec{E}_{ext}$

$N_e \vec{E}_0$

substance	N_2	C_2H_4	C_6H_6	polystyrene	diamond
ε_r	1.0006	2.24	2.28	2.5	5.7

(2) Ionic

$\vec{E}_{ext}$

Cl^- Na^+

$-q\vec{E}_{ext}$ $+q'\vec{E}_{ext}$

substance	KCl	NaCl	$CaCO_3$	PbO
ε_r	4.8	6.0	6.1	26

(3) Orientational (permanent dipoles):

substance	$CHCl_3$	C_6H_5Cl	$C_6H_5NO_2$
ε_r	5.1	10.3	36.5

$\vec{E}_{ext}$ $\vec{p}$ $\vec{\Gamma} = \vec{p} \times \vec{E}_{ext}$

If all microscopic dipole moments (induced or permanent) are identical, $\vec{P} = n\vec{p}$. In the case of permanent dipoles, **p** is fixed, therefore P, and as a result χ and ε_r, are largest when *n* is largest.

(4) H-bond network dynamics (hydrogen-bonded systems – see TOPIC 6):

The dipole moment & intermolecular interactions of water and alcohol is determined basically by the O-H group and by the H-bonds which it forms. P and ε_r are largest for largest density of dipoles and of H-bonds:

| substance	water	methanol	ethanol	1-propanol
molecular formula	H_2O	CH_3OH	$CH_3(CH_2)OH$	$CH_3(CH_2)_2OH$
ε_r	81	34	26	22

Ferroelectric materials & applications

ionic insulators → $\varepsilon_r = 5 \div 10$
ferroelectrics → $\varepsilon_r = 10^3 \div 10^4$

Ferroelectrics: $BaTiO_3$, $PbTiO_3$, $KNbO_3$, $LiNbO_3$
(T_c = 120 °C)

high-T structure:
Perovskite lattice
low-T: Ti^{4+} ion no longer fits in the octahedron of O^{2-} ions & moves to off-centered site
⇒ **dipole moment**

- Ca²⁺ or Ba²⁺ or Pb²⁺ Ca^{2+} or Ba^{2+} or Pb^{2+}
- Ti^{4+}
- O^{2-}

$BaTiO_3$
Cubierta de plata
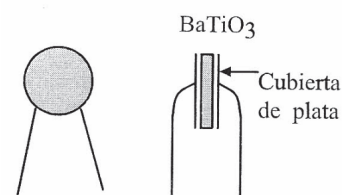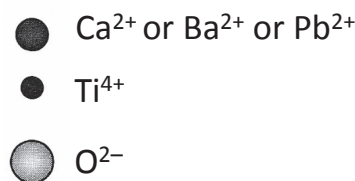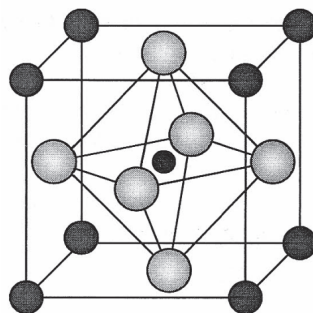
$BaTiO_3$ capacitor, top and side view. It can store 1000 more charge than a normal capacitor

These materials are usually **piezoelectric** :
(used in atomic microscopes, microphones, energy harvesting devices, …)
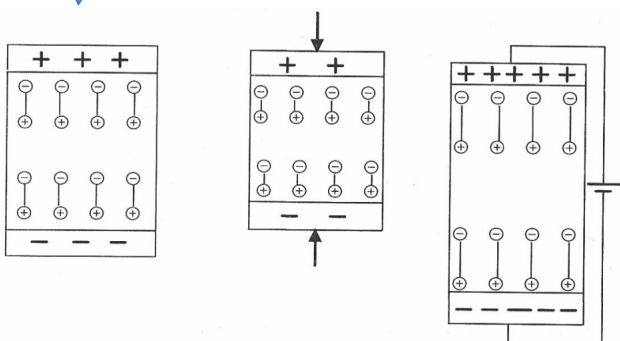
Left: Spontaneous polarization of a ferroelectric crystal
Middle: voltage across material is lowered upon compression
Right: Elongation of material is obtained applying an external potential

Microscopic interactions in dielectrics I

Between charged ions and/or dipolar molecules

- charge-charge (example NaCl):
The force is given by Coulomb's law
interaction energy:

$$U = q_1 V_{q2} = q_2 V_{q1} = \frac{q_1 q_2}{4\pi\varepsilon_o r} \propto \frac{1}{r}$$
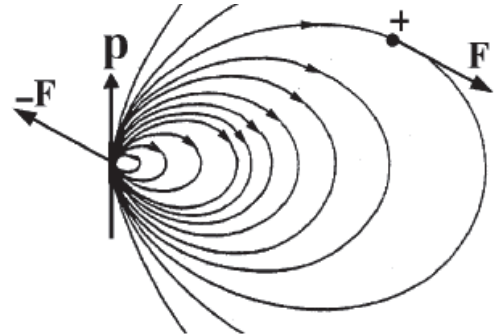
- charge-dipole:
The direction of the force can be found graphically:
(by Newton's 3rd law the forces are equal and opposite)
Interaction energy (both expressions are equal since $\vec{r}' = -\vec{r}$)

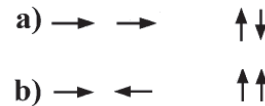$$qV_{\text{dipole}}(\vec{r}) = q\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} = U(\vec{r}) =$$

$$= U(\vec{r}') = -\vec{p}\cdot\vec{E}_{\text{charge}}(\vec{r}') = -\vec{p}\cdot q\frac{\vec{r}'}{4\pi\varepsilon_0\vec{r}'^3} \propto \frac{1}{r^2}$$

- dipole-dipole:
as between 2 charges, **F** is attractive (a) or repulsive (b) :
Interaction energy:

$$U = -\vec{p}_1\cdot\vec{E}_s(r) = -\vec{p}_1\cdot\left[\frac{3(\vec{p}_2\cdot\vec{r})}{4\pi\varepsilon_0}\frac{\vec{r}}{r^5} - \frac{\vec{p}_2}{4\pi\varepsilon_0 r^3}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{\vec{p}_1\cdot\vec{p}_2}{r^3} - \frac{3(\vec{p}_1\cdot\vec{r})(\vec{p}_2\cdot\vec{r})}{r^5}\right] \propto \frac{1}{r^3}$$

Microscopic interactions in dielectrics II

With induced dipoles (apolar molecules) $\mathbf{p} = \alpha\,\mathbf{E}_S$ ($\mathbf{E}_S \rightarrow$ field due to nearby molecule(s))

- charge-induced dipole: $\mathbf{p} \propto \mathbf{E}_S \propto 1/r^2$. The potential due to the induced dipole goes like p/r^2
Hence $U \sim q\,p/r^2 \sim 1/r^4$ and it is attractive: $U = -C_4/r^4$ (with $C_4 > 0$)

- permanent dipole-induced dipole, $\mathbf{p} \propto \mathbf{E}_S \propto 1/r^3$. The potential created by the induced dipole scales as $p/r^2 \sim 1/r^5$, hence its field goes as $\sim 1/r^6$. Since the energy of a dipole in an electric field is the product between p and E, $U = -C_6/r^6$ (with $C_6 > 0$)

- instantaneous dipole-induced dipole: force between 2 neutral apolar molecules!! $U = -C_6/r^6$
Attractive forces between neutral apolar species are called London or Van der Waals forces.
Classically, they arise from fluctuations of the atomic charge resulting in instantaneous dipoles, but they are really a quantum effect, due to vacuum fluctuations of the e.m. field

In all cases, including those of the previous slide, a repulsive potential must be added at short distances, since molecules and atoms cannot overlap (Pauli exclusion principle). The form of the repulsive potential is empirical or chosen for mathematical convenience. An example of potential energy used in van der Waals solids of neutral apolar molecules is the Lennard-Jones potential (the repulsive term is here chosen to simplify calculations):

$$U_{LJ}(r) = \frac{A}{r^{12}} - \frac{B}{r^6}$$

$\rightarrow$ connections with solid state physics course
- Madelung sums in ionic and van der Waals solids $\rightarrow$ COHESIVE ENERGY
- Average effect of interactions $\rightarrow$ ELECTROSTATIC SCREENING. Example: force on a point charge embedded in a dielectric

Magnetic materials and magnets (TOPIC 3)

Importance of SPIN in physics

$$\vec{s} \Rightarrow \begin{cases} \vec{L} = \hbar\vec{s} \\ \vec{m} = \mu_{(\,)}\vec{s} \\ \text{Statistics} \end{cases}$$

| $PARTICLE$	μ
e^-	$\mu_{e^-} = -9.27 \cdot 10^{-24}$
p^+	$\mu_{p^+} = 1.41 \cdot 10^{-26}$
n^0	$\mu_{n^0} = -0.966 \cdot 10^{-26}$

From the magnetic point of view, each spin is a little magnet. Since the electronic spin is the largest one, but also because electron spins are much more correlated than nuclear spins, the magnetic properties of materials basically stem from the magnetic dipole of their electrons. The electronic magnetic moment in an atom (especially for heavy elements, which are those for whic magnetism occurs) is not really proportional to **s**, but to **j** = **s** + **l** , where **l** is the orbital quantum number (this is due to the so-called spin-orbit coupling: see quantum physics course)

We will see soon that a magnetic material can be described as a collection of "spins" (or **j** moments), and as such we will study them in a fashion similar to that employed for polarized dielectric. We will also see that magnetic interactions in a magnet, for example, is really of electrostatic origin. These two characteristics of magnetic materials justify discussing magnetic materials at this point of the course; not to mention that historically the first scientific description of magnetism came well before the description of currents.
Before we go into all that, however, let's start by first looking at the magnetic field of a single electron, and at the effect that a magnetic field has on a single spin.

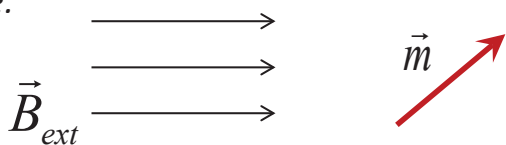# Force, torque & energy for a magnetic dipole

Electric case:
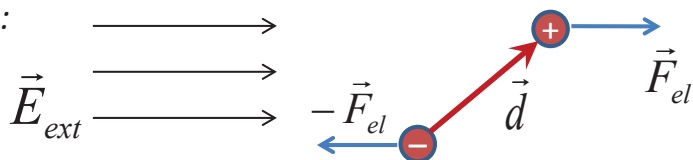
$$\vec{F}_{el} = q\vec{E}_{ext} \quad , \quad \vec{p} = q\vec{d}$$
$$\vec{\Gamma} = \vec{p} \times \vec{E}_{ext}$$
$$U = -\vec{p} \cdot \vec{E}_{ext}$$

Magnetic case:

$$\vec{F}_{mag} = "q_m" \vec{B}_{ext} \quad , \quad \vec{m} = "q_m" \vec{d}$$
$$\vec{\Gamma} = \vec{m} \times \vec{B}_{ext}$$
$$U = -\vec{m} \cdot \vec{B}_{ext}$$

To perform experiments with magnetic poles "q_m", it suffices to take a magnet in the form of a long rod: the north and south poles are then sufficiently far that they can be made to interact separately with other "poles". In this way Coulomb actually demonstrated the existence of a magnetic force between poles that goes like the inverse square of the distance, just like Coulomb's force between point charges. However, while there exist isolated charges (electrons, protons,..), isolated **magnetic monopoles do not exist**: if you break a magnet in two, you get two magnets each with a south and a north pole, so that the net pole is always zero: in this sense magnetic poles display similar features with bound charges in dielectrics.
The fact that there are no magnetic monopoles has a major consequence: electrostatic fields start at positive charges, end at negative charges and are conservative; instead, magnetic fields don't start or end at any point (at most at infinity), since there are no monopoles: **magnetic fields are are solenoidal**. This is the key feature we will exploit to find the field of a magnet.

The magnetic dipole (spin) field

Like an electric (point) dipole produces the E-field $\vec{E}_{\text{point dipole}} = \dfrac{1}{4\pi\varepsilon_0}\left(3\dfrac{\vec{p}\cdot\vec{r}}{r^5}\vec{r} - \dfrac{\vec{p}}{r^3}\right)$, a magnetic dipole (spin) generates the B-field:

$$\vec{B}_{dipole} = \frac{\mu_0}{4\pi}\left[3\frac{\vec{m}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{m}}{r^3}\right]$$

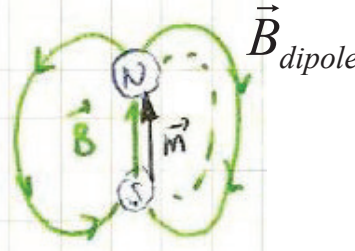Formally, such field is conservative: we can write $\vec{B}_{\substack{magnetic\\dipole}} = -\mu_0\vec{\nabla}\Xi_{\substack{magnetic\\dipole}}$ with $\Xi_{\substack{magnetic\\dipole}} = \dfrac{\vec{m}\cdot\vec{r}}{4\pi r^3}$

However, we know that B is not conservative but rather solenoidal. In fact what happens here is that the singularity at $r = 0$ masks the solenoidal character of the field:
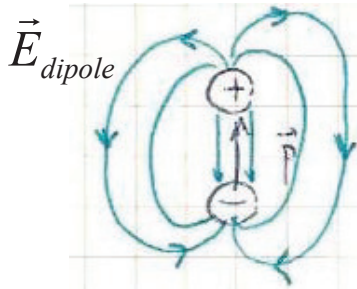
$$\frac{1}{4\pi\varepsilon_0} = 9\cdot 10^9 \text{ [SI]}$$

$$|\vec{p}| \approx e\text{Å} \approx 10^{-30} \text{ [SI]}$$

$$\frac{\mu_0}{4\pi} = 10^{-7} \text{ [SI]}$$

$$m \approx 10^{-23} \text{ [SI]}$$

SI unit of **B**: Tesla (T)
SI unit of μ_0 : T·m/A
$$\left(\oint\vec{B}\cdot d\vec{\ell} = \mu_0 I\right)$$
$\Rightarrow$ SI unit of **m** : A m²

$$\Rightarrow \frac{B}{E} \approx 10^{-10} \quad \text{[SI units]}$$

⇒ **While electric-dipole interactions play a dominant role in polar dielectric materials, magnetic-dipole interactions are too weak to be the source of magnetism (e.g. in iron)**

Vector potential for a magnetic dipole

We want to prove $\vec{B}_{\substack{magnetic\\dipole}} = \vec{\nabla}\times\vec{A}_{\substack{magnetic\\dipole}}$, with $\vec{A}_{\substack{magnetic\\dipole}} = \dfrac{\mu_0}{4\pi}\dfrac{\vec{m}\times\vec{r}}{r^3}$. For this purpose, we need the following vector identity: if $\vec{C}$ does not depend on $\vec{r}$, $\vec{\nabla}\times(\vec{C}\times\vec{G}(\vec{r})) = \vec{C}(\vec{\nabla}\cdot\vec{G}) - (\vec{C}\cdot\vec{\nabla})\vec{G}$

proof:

$$\vec{C}\times\vec{G}(\vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ C_x & C_y & C_z \\ G_x & G_y & G_z \end{vmatrix} \Rightarrow \vec{\nabla}\times(\vec{C}\times\vec{G}(\vec{r})) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ C_yG_z - C_zG_y & C_zG_x - C_xG_z & C_xG_y - C_yG_x \end{vmatrix}$$

$$\left[\vec{\nabla}\times(\vec{C}\times\vec{G}(\vec{r}))\right]_x = C_x\partial_y G_y - C_y\partial_y G_x - C_z\partial_z G_x + C_x\partial_z G_z + \{C_x\partial_x G_x - C_x\partial_x G_x\} =$$

$$= C_x(\partial_x G_x + \partial_y G_y + \partial_z G_z) - (C_x\partial_x + C_y\partial_y + C_z\partial_z)G_x = C_x(\vec{\nabla}\cdot\vec{G}) - (\vec{C}\cdot\vec{\nabla})G_x$$

Proof that B can be obtained from a vector potential: (not at all trivial, only works for $\dfrac{\vec{r}}{r^3}$)

$$\vec{\nabla}\times\vec{A}_{\substack{magnetic\\dipole}} = \vec{\nabla}\times\left(\frac{\mu_0}{4\pi}\vec{m}\times\frac{\vec{r}}{r^3}\right) = \frac{\mu_0}{4\pi}\vec{m}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r^3}\right) - \frac{\mu_0}{4\pi}(\vec{m}\cdot\vec{\nabla})\frac{\vec{r}}{r^3} = \mu_0\delta^3(\vec{r})\vec{m} - \mu_0\vec{\nabla}\left(\underbrace{\frac{\vec{m}\cdot\vec{r}}{4\pi r^3}}_{\Xi_{\substack{magnetic\\dipole}}}\right) =$$

$$= \mu_0\delta^3(\vec{r})\vec{m} + \vec{B}_{\substack{magnetic\\dipole}} \quad \text{Hence for } \vec{r}\neq 0 , \quad \vec{\nabla}\times\vec{A}_{\substack{magnetic\\dipole}} = \vec{B}_{\substack{magnetic\\dipole}}$$

This further shows that B is solenoidal, since $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

(in fact $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{G}) = 0 \quad \forall\vec{G}$, see problem 0-9)

Magnetized media

A magnetized medium is a collection of magnetic dipoles. We cannot deal with all magnetic dipoles in a sample (the total number of spins is that of unpaired electrons, which is of the order of the number of atoms, or Avogadro's number). We therefore define a MAGNETIZATON FIELD as the average magnetic dipole moment per unit volume:

Magnetization field
$$\vec{M} = \frac{1}{\delta\tau}\sum_i^{\delta N} \vec{m}_i$$

Hence an elementary volume $d\tau$ of a magnetized sample carries an elementary magnetic dipole moment equal to: $d\vec{m} = \vec{M}d\tau$. Just as we did in the case of polarized dielectrics in topic 2, let us define the POLE DENSITIES associated with the magnetization field, as:

$$\begin{cases} \rho_m = -\vec{\nabla}\cdot\vec{M} & \text{Volume POLE DENSITY} \\ \sigma_m = \vec{M}\cdot\hat{n} & \text{Surface POLE DENSITY} \end{cases}$$

The sign of the magnetic poles can be positive (north pole) or negative (south pole). (A magnetic dipole moment goes from south to north, that is, from − to + like an electric dipole). These definitions are consistent with our experience that a bar magnet has a south and a north pole.

SI unit of **m** : A·m² ⇒ unit of **M** and σ_m : A/m ¿Why does the ampere (A) appear here?→ topic 4

What is the B-field produced by a magnetized object such as a magnet? We cannot calculate it using the pole density as source of B as the bound charge was the source of E, because otherwise we would get a conservative B that starts at north poles and ends at south poles. This would contradict the solenoidal nature of B: the field lines of B go around in closed loops, they never start nor end. The way to get a solenoidal field is to start with the vector potential **A** of the magnet and then taking its curl: $\vec{B}_{magnet} = \vec{\nabla}\times\vec{A}_{magnet}$. This guarantees that $\vec{\nabla}\cdot\vec{B}_{magnet} = 0$

Fundamental theorem of magnetized media

We know that for a single magnetic dipole the vector potential is $\vec{A}_{magnetic\ dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$

Hence for a continuous collection of dipoles (magnetized object) each at position **r'** one has:

$$\vec{A}(\vec{r}) = \int d\vec{A} = \frac{\mu_0}{4\pi}\int_{Vol} d\vec{m}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0}{4\pi}\int_{Vol} d\tau'\,\vec{M}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$. Therefore, taking the curl:

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A} = \frac{\mu_0}{4\pi}\int d\tau'\,\vec{\nabla}\times\left(\vec{M}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = \frac{\mu_0}{4\pi}\int d\tau'\left[\vec{M}(\vec{r}')\vec{\nabla}\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} - (\vec{M}\cdot\vec{\nabla})\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right]$$

$$= \frac{\mu_0}{4\pi}\int d\tau'\,\vec{M}(\vec{r}')4\pi\delta^3(\vec{r}-\vec{r}') - \frac{\mu_0}{4\pi}\int d\tau'\,\vec{\nabla}\left(\vec{M}\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\right) = \mu_0\vec{M}(\vec{r}) + \frac{\mu_0}{4\pi}\cdot(-\vec{\nabla})\int d\tau'\,\vec{M}(\vec{r}')\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

We have obtained: $\vec{B} = \mu_0\vec{M} - \mu_0\vec{\nabla}\Xi$, where $\Xi(\vec{r}) = \frac{1}{4\pi}\int d\tau'\,\vec{M}(\vec{r}')\cdot\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$ is the total scalar potential of the magnetized object.

Integrating by parts, it is straightforward to show (see fundamental theorem of polarized media, topic 2) that the scalar potential can be re-written as:

$$\Xi(\vec{r}) = \frac{1}{4\pi}\oint_S \frac{\vec{M}(\vec{r}')\cdot\hat{n}}{|\vec{r}'-\vec{r}|}da + \frac{1}{4\pi}\int_V d\tau\frac{-\vec{\nabla}'\cdot\vec{M}(\vec{r}')}{|\vec{r}'-\vec{r}|} = \frac{1}{4\pi}\oint_S \frac{\sigma_m}{|\vec{r}'-\vec{r}|}da + \frac{1}{4\pi}\int_V \frac{\rho_m}{|\vec{r}'-\vec{r}|}d\tau$$

Here we have used the previously given definitions of surface and volume pole densities. Thus **B** can be obtained indirectly with the help of a conservative "auxiliary" field **H**:

$$\boxed{\vec{B} = \mu_0\left(\vec{M}+\vec{H}\right) \quad \text{with} \quad \vec{H}(\vec{r}) = -\vec{\nabla}\Xi = \frac{1}{4\pi}\int d\tau'\rho_m(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \frac{1}{4\pi}\int da'\,\sigma_m(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}$$

Problems with magnetized media

Strategy:

(1) from **M**, find the magnetic pole densities $\rho_m = -\vec{\nabla} \cdot \vec{M}$ and $\sigma_m = \vec{M} \cdot \hat{n}$

(2) from ρ_m and σ_m, calculate **H** as you would calculate **E** (without the factor ε_0), using:

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int d\tau' \rho_m(\vec{r}') \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \frac{1}{4\pi}\int da' \sigma_m(\vec{r}') \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

(or calculate Ξ as you would calculate V, and then compute the auxiliary field as $\vec{H} = -\vec{\nabla}\Xi$)

(3) From **M** and **H**, calculate **B** as $\vec{B} = \mu_0(\vec{H} + \vec{M})$

Example: **bar magnet**

the magnetic pole density consists of two disks of uniform charge density at both ends of the rod. The H field has the same form as the E-field produced by two charged plates. The B field, which is what we want to get as it determines magnetic forces, is then found as $\vec{B} = \mu_0(\vec{H} + \vec{M})$

→ the black lines are the lines of H (or E)
The white lines inside the rod and the black lines outside it are the lines of B (or D) (outside, that is, in vacuum, $\vec{B} = \mu_0 \vec{H}$)
Notice how the **LINES of B FORM CLOSED LOOPS**

SI unit of **m** : A·m² $\Rightarrow$
SI unit of **M**, **H** and σ_m : A/m
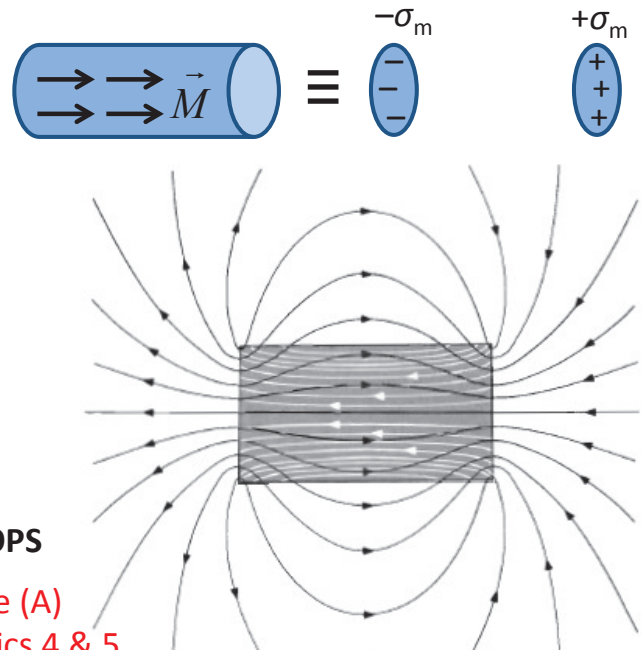
¿Why does the Ampere (A) appear here?? see topics 4 & 5

Application: B-field of a spherical magnet

Spherical magnet = uniformly magnetized sphere

Since M is uniform, the volume pole density is zero: $\rho_m = -\vec{\nabla} \cdot \vec{M} = 0$

The surface pole density of a uniformly magnetized sphere is: $\sigma_m(r) = \hat{n} \cdot \vec{M}(r) = \vec{M}\cos\theta$

It acts as source of H (equivalent electrostatics problem: uniformly polarized sphere)

We already know how to calculate the (H) field of a surface density proportional to $\cos\theta$:

- outside the sphere the field H is a dipolar field $\vec{H}(\vec{r}) = \frac{1}{4\pi}\left(\frac{3\vec{m}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{m}}{r^3}\right)$

with the magnetic moment $\vec{m} = \vec{M}\,Vol$ placed in the sphere's center

- inside the field H is uniform (as the E field inside a polarized sphere) and equal to $\vec{H} = -\frac{\vec{M}}{3}$

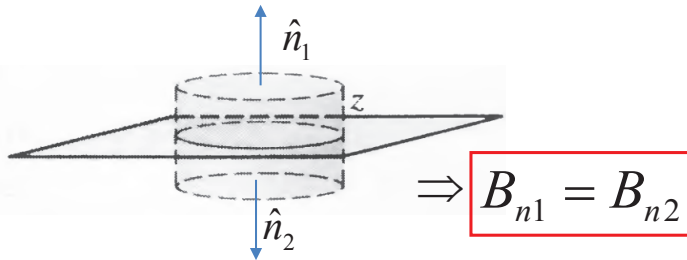The magnetic field B (which is what we are interested in) is then:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{m}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{m}}{r^3}\right) \text{ outside} \; ; \; \vec{B} = \frac{2}{3}\mu_0\vec{M} \text{ inside}$$
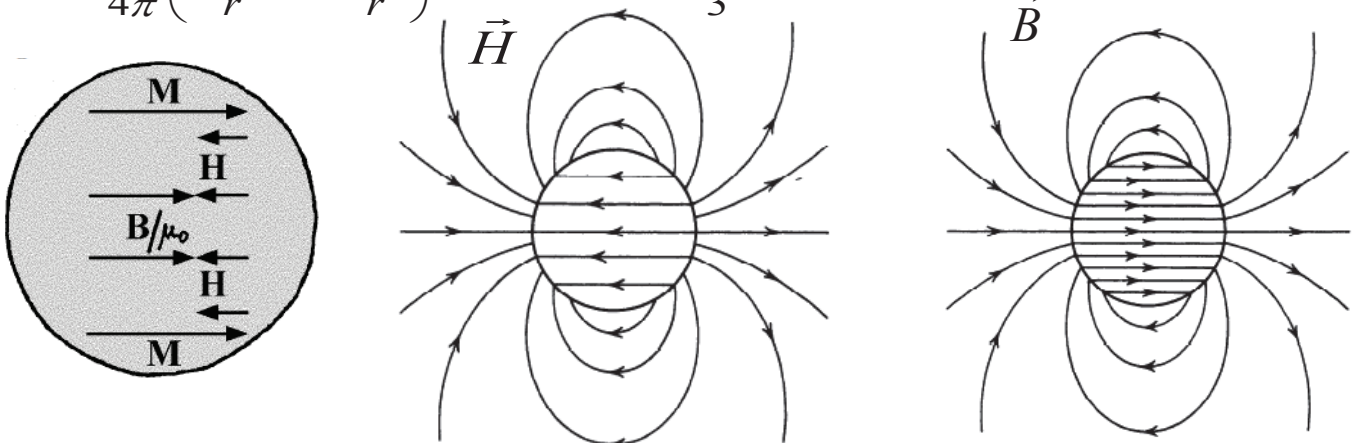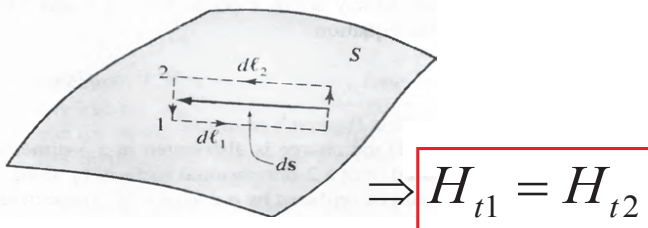
Field equations and boundary conditions

Field equations for the auxiliary H field:

B is solenoidal : $\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow$

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} = \rho_m \\ \vec{H} = -\vec{\nabla} \Xi_m \Rightarrow \vec{\nabla} \times \vec{H} = 0 \end{cases}$$

$$\Phi(\vec{B}) = \vec{B}_1 \cdot \hat{n}_1 \, Area + \vec{B}_2 \cdot \hat{n}_2 \, Area = 0$$

$$\vec{\nabla} \cdot \vec{H} = \rho_m \quad \Rightarrow \quad \Phi(\vec{H}) = "q_m"$$

$$\Phi(\vec{H}) = \vec{H}_1 \cdot \hat{n}_1 \, Area + \vec{H}_2 \cdot \hat{n}_2 \, Area$$
$$= "q_m" = \sigma_m \, Area$$

$$\Rightarrow \boxed{B_{n1} = B_{n2}}$$

$$\vec{\nabla} \times \vec{H} = 0 \quad \Rightarrow$$

$$H_{t2} d\ell_2 + H_{t1} d\ell_1 = (H_{t2} - H_{t1})|d\ell| = 0$$

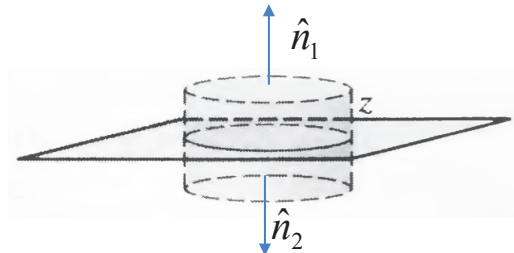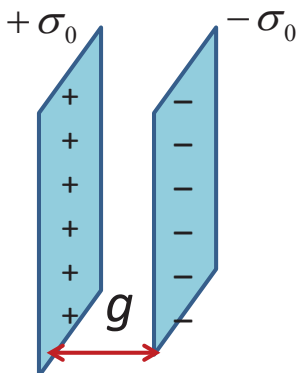$$\Rightarrow \boxed{H_{t1} = H_{t2}}$$

$$\Rightarrow \boxed{\vec{H}_1 \cdot \hat{n}_1 + \vec{H}_2 \cdot \hat{n}_2 = \sigma_m}$$

Application of boundary conditions
→ *midterm exam 2013*

Permanent-magnet magnetic circuit
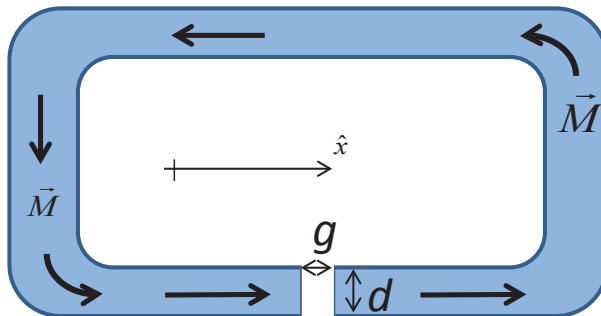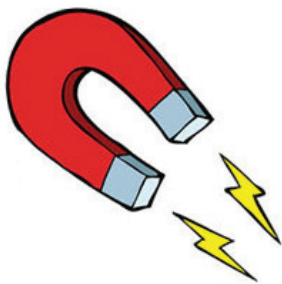
If the magnitude of **M** is constant, and the field lines of **M** are always parallel to the sides of the magnet, then there are no volume poles: $\rho_m = -\vec{\nabla} \cdot \vec{M} = 0$. The only magnetic pole density is a surface density, present at the air gap: $\sigma_m = \vec{M} \cdot \hat{n} = \pm |\vec{M}|$

This surface pole density acts as source of H as a charge density of identical geometry, such as the one present in a parallel-plate capacitor, generates an electrostatic field E. In the limit that the air gap size g is much smaller than the linear dimension d of the magnet's cross section, the planar symmetry approximation can be applied, so we have:

$$\vec{H} = \begin{cases} \sigma_m \hat{x} = |\vec{M}|\hat{x} & \text{, in the air gap} \\ 0 & \text{, in the magnet} \end{cases}$$

Therefore:

$$\vec{B} = \mu_0(\vec{M} + \vec{H}) = \begin{cases} \mu_0 \vec{M} & \text{, in the magnet} \\ \mu_0 \vec{H} = \mu_0 |\vec{M}|\hat{x} & \text{, in the gap} \end{cases} = \mu_0 \sigma_m \hat{x} \Rightarrow$$

B is the same inside and outside the magnet: $\vec{B} = \mu_0 \sigma_m \hat{x}$

Magnetic force at the air gap
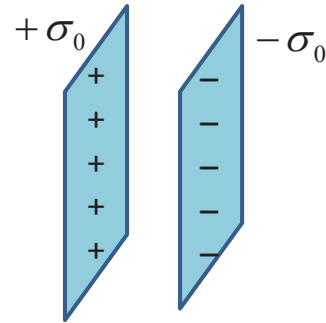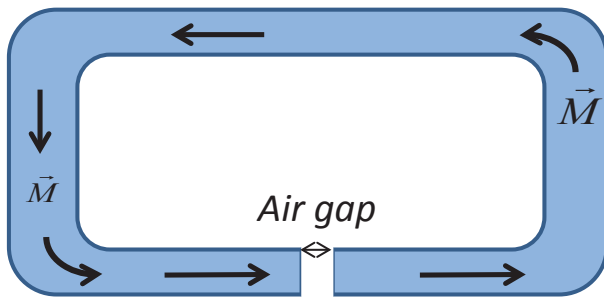
We know from Coulomb's experiments that a magnetic field exerts a force on a magnetic pole, given by $\vec{F}_{mag} = "q_m" \vec{B}_{ext}$, much in the same way as there is a force due to an electric field applied to a charge: $\vec{F}_{el} = q \vec{E}_{ext}$. In the case of a parallel-plate capacitor, the force on a metal plate is given by

$$F_{el} = \frac{1}{2} q_f E = \frac{1}{2} \sigma_f \, Area \, E$$

Here E is the total electric field in the capacitor, and the factor ½ is introduced to avoid including the field due to the charge on which the force is exerted. We see that the electrostatic pressure pushing the plates together is equal to the electrostatic energy density $p_{el} = \frac{1}{2} \sigma_f E = \frac{1}{2} D \cdot E$
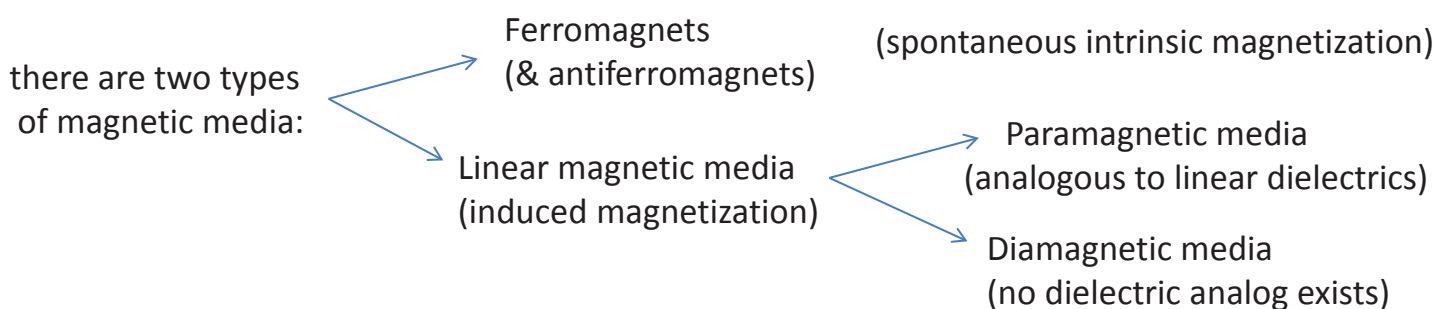
One could argue that the same result should apply to the air gap in a magnetic circuit, namely:

$$F_{mag} = \frac{1}{2} \sigma_m \, Area \, B \Rightarrow p_{mag} = \frac{1}{2} \sigma_m B = \frac{1}{2} H \cdot B$$

We will see in topic 5 that $\frac{1}{2} H \cdot B$ is indeed the energy density associated with magnetic fields.

for the class: ¿Why didn't we consider the simpler case of a uniformly magnetized thin slab?

Magnetic materials

there are two types of magnetic media:

Ferromagnets (& antiferromagnets) — (spontaneous intrinsic magnetization)

Linear magnetic media (induced magnetization)

Paramagnetic media (analogous to linear dielectrics)

Diamagnetic media (no dielectric analog exists)

Most materials are non-magnetic or show only weak magnetism as a (linear) response to an applied field. Few materials, for example magnetite Fe_3O_4, show strong magnetic properties: these materials are the ones used as permanent magnets (in Spanish, "imanes")

In the case of linear magnetic material, we will be interested in the magnetization that arises when a linear magnetic material is placed in an applied field. A linear magnetic material is one in which the magnetization is directly proportional to the <u>macroscopic</u> field inside the sample. To keep the analogy that we have been using, a linear magnetic medium is such that: $\vec{M} = \chi_m \vec{H}$

The constant of proportionality is a pure number, χ_m , called **MAGNETIC SUSCEPTIBILITY**
Since $\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$, we then get $\vec{B} = (1 + \chi_m) \mu_0 \vec{H} = \mu_r \mu_0 \vec{H}$, where we defined the **RELATIVE PERMITTIVITY** as $\mu_r = 1 + \chi_m$

As for the electric case, these equations are only apparently simple, because the macroscopic field H is itself an unknown and depends on M.

Analogy of magnetic & dielectric media

Notice the analogies
Between the **B** and **D**
field equations

$$\begin{cases} \vec{D} = \varepsilon_0\vec{E} + \vec{P} \\ \vec{B} = \mu_0\vec{H} + \mu_0\vec{M} \end{cases} \qquad \begin{cases} \vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P} & \vec{\nabla}\cdot\vec{D} = \rho_f \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{\nabla}\times\vec{M} & \vec{\nabla}\cdot\vec{B} = 0 \end{cases}$$

Mathematical
equivalence of the
H field produced by
magnetized bodies
and the **E** field:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau'\rho(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \frac{1}{4\pi\varepsilon_0}\int da'\sigma(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad,\quad \vec{\nabla}\times\vec{E} = 0$$

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int d\tau'\rho_m(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} + \frac{1}{4\pi}\int da'\sigma_m(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \quad,\quad \vec{\nabla}\times\vec{H} = 0$$

Note also the analogies
between the equations
for linear materials:

$$\vec{P} = \chi_e\varepsilon_0\vec{E} = (\varepsilon_r-1)\varepsilon_0\vec{E} \qquad \vec{D} = (\chi_e+1)\varepsilon_0\vec{E} = \varepsilon_r\varepsilon_0\vec{E}$$

$$\vec{M} = \chi_m\vec{H} = (\mu_r-1)\vec{H} \qquad \vec{B} = (1+\chi_m)\,\mu_0\vec{H} = \mu_r\mu_0\vec{H}$$

Problems with magnetized media can be solved as those with polarized media by substituting:

$$\boxed{\frac{\vec{B}}{\mu_0} \leftrightarrow \vec{D} \;;\quad \vec{H} \leftrightarrow \varepsilon_0\vec{E} \;;\quad \vec{M} \leftrightarrow \vec{P} \;;\quad \rho_m \leftrightarrow \rho_b \;;\quad 0 \leftrightarrow \rho_f}$$

The similar effect that E and B have on dipoles (whether electric or magnetic), together with the analogy between magnetized and polarized systems, suggest that we may also write the energy of a set of linear magnetic objects in a magnetic field as:

$$U_{mag} = \int d\tau\,\frac{1}{2}\vec{B}\cdot\vec{H} = \int d\tau\,\frac{B^2}{2\mu_0\mu_r}$$

Linear magnetic materials

$$\vec{M} = \chi_m\vec{H} = (\mu_r-1)\vec{H} \qquad\qquad \vec{B} = (1+\chi_m)\,\mu_0\vec{H} = \mu_r\mu_0\vec{H}$$

As for the dielectric case, these equations are only apparently simple, because the macroscopic field H (which is a priori unknown) depends on M.

Example: a sphere of linear magnetic material in an otherwise uniform external field B$_{ext}$. Since in vacuum B = μ$_0$H, this is the same as saying that the sphere is in uniform applied H$_{ext}$ = B$_{ext}$/μ$_0$. To find M as function of H$_{ext}$, we assume that the induced magnetization is parallel to H$_{ext}$ and constant . The magnetization creates a surface magnetic pole density which itself generates an H-field which may be calculated from the pole density, proportional to cosθ. We get:

$$\vec{M} = \chi_m\left(\vec{H}_{ext} + \vec{H}_{demagnetising}\right) = \chi_m\vec{H}_{ext} - \chi_m\frac{\vec{M}}{3}$$. Hence $M(1+\chi_m/3) = \chi_m\vec{H}_{ext}$, or:

$$\vec{M} = \frac{3\chi_m}{3+\chi_m}\vec{H}_{ext} = 3\frac{\mu_r-1}{\mu_r+2}\vec{H}_{ext}$$. Hence B outside is μ$_0$H$_{ext}$ plus a dipolar field, while inside

$$\vec{B}_{inside} = \mu_0\left(\vec{H}+\vec{M}\right) = \mu_0\left(\vec{H}_{ext}+\vec{H}_{magnetising}+\vec{M}\right) = \mu_0\left(\vec{H}_{ext}+\frac{2}{3}\vec{M}\right) = 3\frac{\mu_r\mu_0}{\mu_r+2}\vec{H}_{ext}$$

$$\boxed{\text{Paramagnetism} \rightarrow \chi_m > 0,\ \mu_r > 1 \qquad \text{Diamagnetism} \rightarrow \chi_m < 0,\ \mu_r < 1}$$

In many cases (especially for diamagnetic media, but also for several paramagnetic materials), χ_m is very small, so that basically μ$_r$ ≈ 1

Diamagnetism *vs* paramagnetism

Origin of paramagnetism ($\chi_m > 0$) → alignment of unpaired electron spins (unpaired magnetic dipoles) of medium with applied field → *equivalent to orientational polarization in dielectrics*

Origin of diamagnetism ($\chi_m < 0$) → response of electron pairs with total spin zero to an applied field : it can be thought to be due to Faraday & Lenz's laws at the atomic scale (see topic 4) → *there exists no dielectric equivalent of diamagnetic materials*

Magnetic susceptibility and permeability: some examples

Paramagnetic substance	χ_m	diamagnetic substance	χ_m	compound	χ_m
O_2 (STP)	$+1.9 \times 10^{-6}$	H_2 (STP)	-2.2×10^{-9}	$H_2O(\ell, 293\,K)$	-9.0×10^{-6}
Na	$+8.5 \times 10^{-6}$	He (STP)	-1.1×10^{-9}	CO (STP)	-5.5×10^{-9}
Al	$+2.1 \times 10^{-5}$	N_2 (STP)	-6.7×10^{-9}	NO (STP)	$+8.2 \times 10^{-7}$
K	$+5.7 \times 10^{-6}$	Si	-3.3×10^{-6}	CO_2 (STP)	-1.2×10^{-8}
Cr	$+2.9 \times 10^{-4}$	Ar (STP)	-1.1×10^{-8}	SiO_2	-1.4×10^{-5}
Rb	$+3.7 \times 10^{-6}$	Cu	-9.6×10^{-6}		
W	$+7.0 \times 10^{-5}$	Xe (STP)	-2.6×10^{-8}		
Nd	$+2.8 \times 10^{-4}$	Au	-3.4×10^{-5}		
Gd	$+8.7 \times 10^{-3}$	Pb	-1.6×10^{-5}		

χ_m is usually quite small, so that basically $\mu_r \approx 1$

Question for the class: how will a paramagnetic or diamagnetic bar orient in an applied field, and why? →


diamagnetic needle   paramagnetic needle
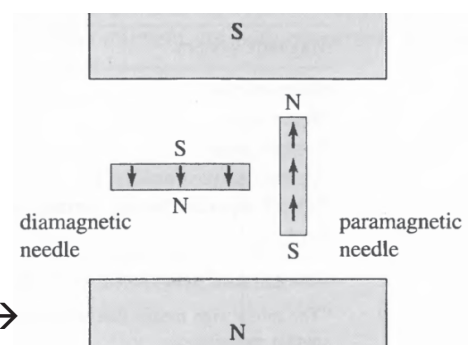
Saturation magnetization
(paramagnetic & ferromagnetic materials)

The value of the microscopic dipole moment is fixed, for a given material, and determined by the electronic orbital configuration: m is proportional to $j = s + l$ (total moment = spin+ orbital moment, see Quantum Physics course, spin-orbit coupling), which is fixed for a given electron. (notice that this is different for the electric case, where there is in principle no theoretical upper limit for the value of $p_{induced}$ as one can pull two bound charges further and further apart)

For a ferromagnetic material, the maximum total magnetization is when all magnetic dipoles in the sample are aligned in the same direction.

For a paramagnetic material the linear relationship between M and H can't hold for large H, since there is a maximum possible magnetization, corresponding to all magnetic moments aligned in the same direction. This implies that the magnetization curve is linear at low H, and then levels off at high H (high applied B field):
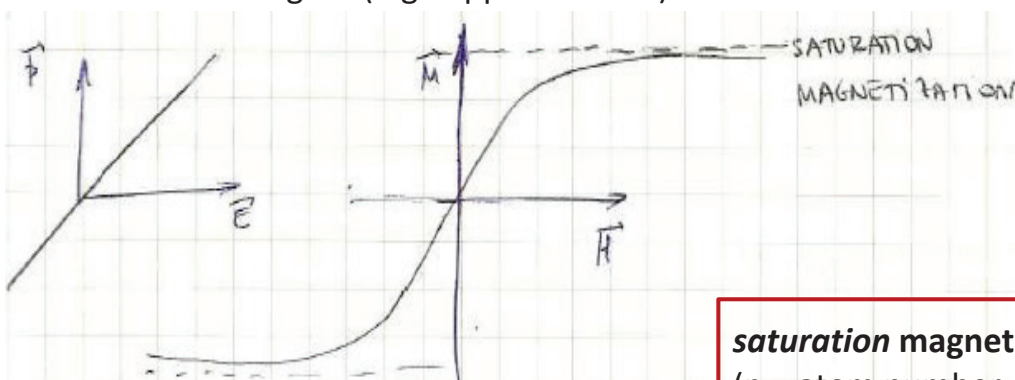
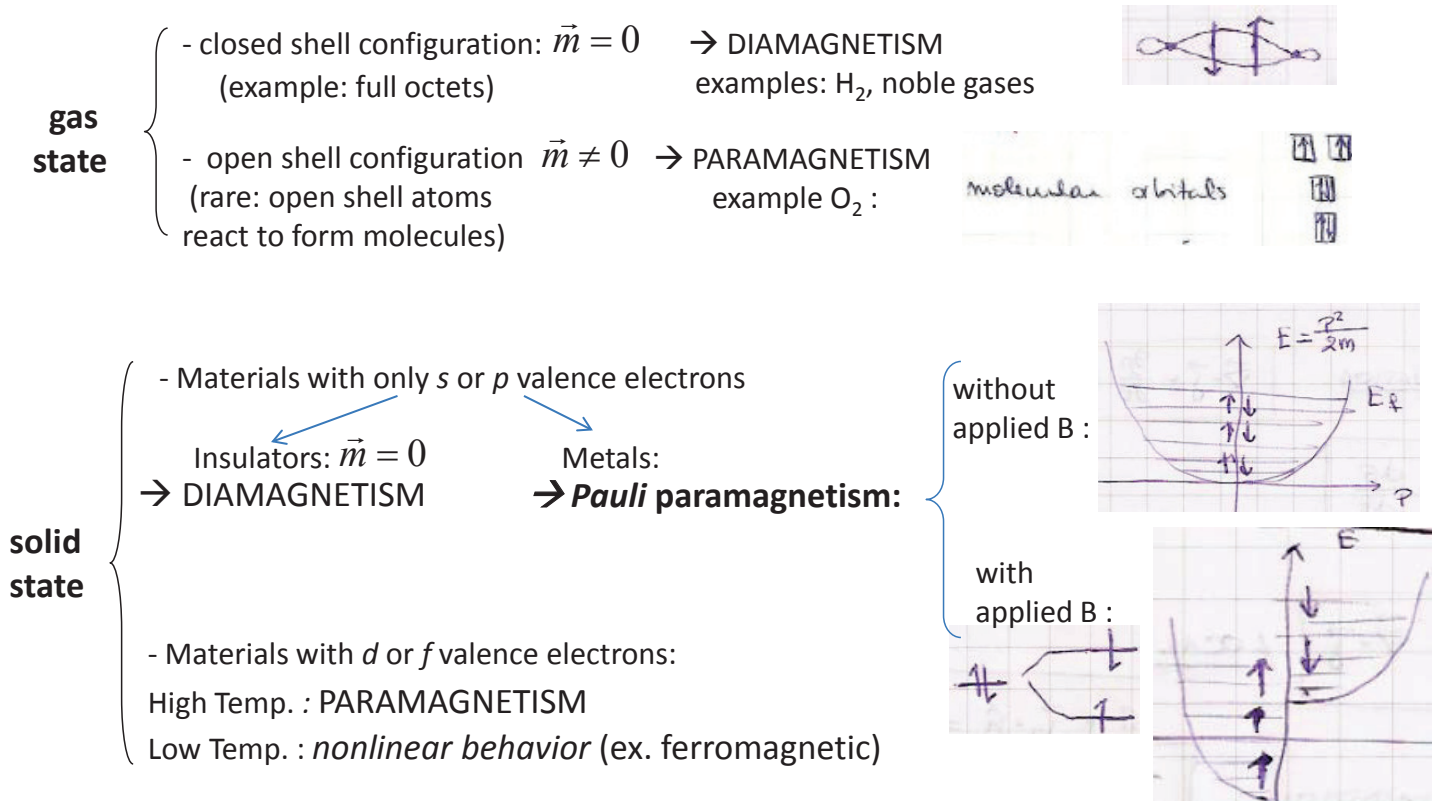paramagnetic materials *are different from linear dielectrics*, in which there is no theoretical upper limit for the induced polarization field **P**

saturation magnetization:
(n = atom number density) $\vec{M}_{sat} = n\,\vec{m}$

Gas- *vs* condensed-phase magnetic media

The class of (non)linear magnetic media is more varied than that of (non)linear dielectrics. Among linear media, there are materials with $\chi_m > 0$ $(\mu_r < 1)$, that are called **paramagnetic** and that are analogous to linear dielectrics; but there are also materials with $\chi_m > 0$ $(\mu_r < 1)$, called **diamagnetic**, and whose magnetization is opposite to that of the applied field

gas state
- closed shell configuration: $\vec{m} = 0$ → DIAMAGNETISM
  (example: full octets)   examples: H_2, noble gases

- open shell configuration $\vec{m} \neq 0$ → PARAMAGNETISM
  (rare: open shell atoms   example O_2 :
  react to form molecules)

solid state
- Materials with only *s* or *p* valence electrons

  Insulators: $\vec{m} = 0$ Metals:
  → DIAMAGNETISM   → *Pauli* paramagnetism:

  without applied B :

  with applied B :

- Materials with *d* or *f* valence electrons:
  High Temp. : PARAMAGNETISM
  Low Temp. : *nonlinear behavior* (ex. ferromagnetic)

Microscopic magnetic interactions

Nonlinear magnetic media: Just like the class of linear magnetic media is more varied than that of linear dielectrics, the behavior of nonlinear magnetic media displaying spontaneous magnetic ordering is also much richer than that of ferroelectric materials.

- Mechanism of interaction: exchange coupling →

- Magnetic behavior: transition from a high temperature phase (usually paramagnetic) to a low-temperature phase with long range magnetic correlations (ferromagnetic, antiferrom., ferrimagnetic, spinel order, Kondo state, incommensurate phases...

phase transitions (e.g. para-ferro)
→ link with statistical physics course