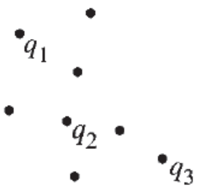
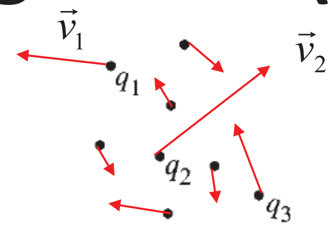
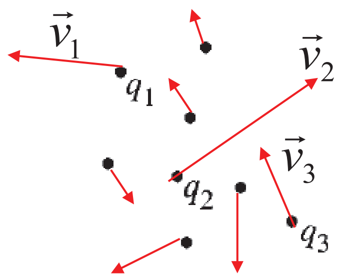
Steady currents & magnetism (TOPIC 4)

electrostatics  vs  electrodynamics

All phenomena described so far concerned charges at rest; we now move on to charges in motion. The situation is considerably more intricate than in the electrostatic case, since a moving charge is: *a*) source of a time-dependent E-field and *b*) source of a magnetic field B. Moreover, due to the Lorentz force it is: *c*) subject to E-fields and *d*) it is also subject to B-fields (if the charge also possesses a spin, also *e*) the effect of B on the spin must be considered) Given the complexity of the task, in topic 4 we deal only with magnetostatics, that is, with steady currents, and we'll see the effect of time-varying charges and currents in topic 5. In this topic, we start with the definition of *currents* and discuss how they are produced by electrochemical devices such as batteries. Ohm's law is used to calculate resistances and current distributions in homogeneous and heterogeneous media. We describe how currents act as *sources of B*, comparing them with magnets. Contents of topic 4:

- *I. Definition of current and current density, charge conservation, boundary condition for **J***
- *II. Batteries and fuel cells*
- *III. Ohm's local law and calculations of resistance ; Joule's law*
- *IV. B- and H-fields generated by steady currents inside or near magnetic materials*
- *V. Magnetic circuits*
- *VI. Ampère's equivalence theorem: superconductors ; magnetic dipole of current loops*

Current density



We define the **current density** as $\vec{J} = \dfrac{\sum_i^{\delta N} q_i \vec{v}_i}{\delta V}$

The vector **J** is a macroscopic field like **P** or **M**. If there is only one type of moving charges, for example electrons (as in a metal wire carrying an electric current), then $q_i = -e \quad \forall i$, which entails:

$$\vec{J} = -e\frac{\sum_i^{\delta N} \vec{v}_i}{\delta V} = -e\frac{\delta N}{\delta V}\frac{\sum_i^{\delta N} \vec{v}_i}{\delta N} = -en\langle\vec{v}\rangle = \rho\langle\vec{v}\rangle$$

The average velocity $\langle\vec{v}\rangle = \dfrac{\sum_i^{\delta N} \vec{v}_i}{\delta N}$ is called ***drift velocity*** . If all charges are the same:

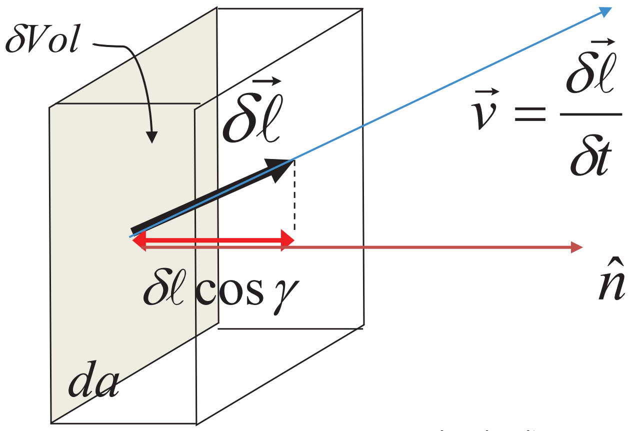
$$\vec{J} = \frac{\sum_i^{\delta N} q\vec{v}_i}{\delta V} = nq\vec{v}_{drift} \qquad \text{or} \qquad \boxed{\vec{J} = \rho\vec{v}_{drift}}$$

If there is more than one type of charge carrier, each having a charge q_α , we average (sum) separately over each kind of particle α : $\vec{J} = \dfrac{\sum_\alpha \sum_{i_\alpha}^{\delta N_\alpha} q_\alpha \vec{v}_{i_\alpha}}{\delta V} = \sum_\alpha q_\alpha n_\alpha \langle\vec{v}_\alpha\rangle = \sum_\alpha \rho_\alpha \langle\vec{v}_\alpha\rangle$

Current: flux of $\vec{J}$

The flux of $\vec{J} = \rho\langle\vec{v}\rangle$ through an infinitesimal surface is (dropping the average sign for clarity):

$$\vec{J}\cdot d\vec{a} = \rho\vec{v}\cdot d\vec{a} = \rho\frac{\delta\vec{\ell}}{\delta t}\cdot d\vec{a} = \frac{\rho\, da\,\delta\ell\cos\gamma}{\delta t} = \frac{\rho\,\delta Vol}{\delta t} = \frac{\delta q_{across}}{\delta t} = i_{across}$$



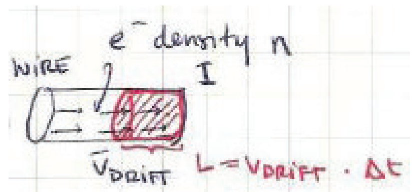
$\Rightarrow$ **J** · **da** is equal to the (average) charge δq crossing the area *da* in a time δt, which is by definition the ***current*** through *da*. Integrating over a larger surface *S*, we get that the flux of **J** through *S* is equal to the total charge dq crossing *S* in time *dt*:

$$I = \frac{dq_{across}}{dt} = \Phi_{\vec{J}} = \int_S \vec{J}\cdot d\vec{a}$$

Example: current in a wire

number density of electrons

$$I = \frac{\Delta Q}{\Delta t} = \rho\frac{\Delta Vol}{\Delta t} = ne\frac{S\Delta L}{\Delta t} = ne\frac{Sv_{drift}\Delta t}{\Delta t} = neSv_{drift} = J_f S$$



Units of I: Ampère (A)

Units of J: A/m²

Copper has one conduction electron per atom and 8.5×10^{28} atoms per cubic meter. Hence $n = 8.5\times10^{28}$ electrons per m³. For a current of 1 ampere in a wire having a cross-section of 1 mm², we have J = 10^6 amperes per square meter, so:

Drift velocity in a common wire

$$v_{drift} = \frac{10^6}{8.5\cdot10^{28}\times1.6\cdot10^{-19}} \approx 7.4\cdot10^{-5}\ \mathrm{m/s}$$

This is about 26 cm per hour !!!

Local charge conservation law

The flux of **J** through a closed surface *S* is equal to the total charge dq_{across} that crosses *S* in a time dt:

$$I = \frac{dq_{across}}{dt} = \Phi_{\vec{J}} = \oint_S \vec{J}\cdot d\vec{a}$$

If the volume enclosed by the surface *S* is equal to *Vol*, we get (using Gauss' integral theorem in the last step):

$$I = \frac{dq_{across}}{dt} = \oint_S \vec{J}\cdot d\vec{a} = \int_{Vol} d\tau\,\vec{\nabla}\cdot\vec{J}$$

global *charge conservation* implies that the charge crossing the surface must equal the charge leaving the volume it encloses:

$$I_{\text{across } S} = -\frac{dQ_{inside}}{dt}$$

As the total charge inside a volume is $Q_{inside} = \int_{Vol} d\tau\,\rho$, this implies:

$$I = \frac{dq_{across}}{dt} = \oint_S \vec{J}\cdot d\vec{a} = \underbrace{\int_{Vol} d\tau\,\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\int_{Vol} d\tau\,\rho}_{} = -\frac{dQ_{inside}}{dt}$$

since this holds for an arbitrary volume *Vol* $\Rightarrow$

$$\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

LOCAL law of charge conservation

→ Charge conservation law for magnetostatics (NO CHARGE ACCUMULATION): $\boxed{\vec{\nabla}\cdot\vec{J} = 0}$

Hence, in magnetostatics, **J** is solenoidal like **B**. From this one obtains the following

→ boundary condition: $\boxed{\oint_S \vec{J}\cdot d\vec{a} = 0 \Rightarrow \vec{J}_1\cdot\hat{n} = \vec{J}_2\cdot\hat{n}}$ (only valid in magnetostatics)

Currents of free and bound charges

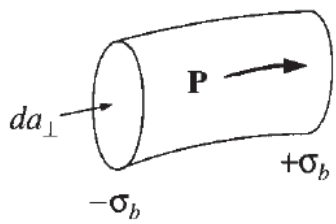
Since we distinguished between two types of charges, free and bound, and since under normal circumstances they cannot convert into each other, free charges and bound charges should be conserved separately: $\vec{J}_{tot} = \vec{J}_f + \vec{J}_b$ with $\vec{\nabla}\cdot\vec{J}_f + \partial_t \rho_f = 0$ and $\vec{\nabla}\cdot\vec{J}_b + \partial_t \rho_b = 0$

Here we have defined the free and bound current densities $\vec{J}_f$ and $\vec{J}_b$. While $\vec{J}_f$ is due to free charges which are able to move freely across macroscopic distances in a sample, bound charge is basically atomic or molecular in nature, and can only displace inside a single molecule. Moreover, when a static electric field is applied, a static polarization results, so that at equilibrium no motion of bound charges takes place. In other words,

$\Rightarrow$ **in steady state conditions** $\vec{J}_b = 0$, $\partial_t \rho_f = 0$ **so that:**

$$\vec{J}_{tot} = \vec{J}_f \quad \text{with} \quad \vec{\nabla}\cdot\vec{J}_f = 0$$
$$\vec{J}_{f,1}\cdot\hat{n} = \vec{J}_{f,2}\cdot\hat{n}$$

There can only be a bound charge current when the polarization is being created or altered. To see what the bound current density is equal to, consider a tiny cylindrical chunk of polarized material. The polarization introduces a bound charge density $\sigma_b = \vec{P}\cdot\hat{n}$ at one end and $-\sigma_b$ at the other. If *P* increases, the bound charge densities increase accordingly, giving a net bound current:



$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial\left(\vec{P}\cdot\hat{n}\right)}{\partial t} da_\perp = \frac{\partial \vec{P}}{\partial t}\cdot d\vec{a}$$

This gives for the bound charge current density: $$\vec{J}_b = \frac{\partial \vec{P}}{\partial t} \qquad (= 0 \ \text{if}\ \partial_t = 0)$$

(this is consistent with charge conservation: since $\rho_b = -\vec{\nabla}\cdot\vec{P}$, we have: $\vec{\nabla}\cdot\vec{J}_b = \frac{\partial \vec{\nabla}\cdot\vec{P}}{\partial t} = -\frac{\partial \rho_b}{\partial t}$)

The "local" form of Ohm's law

$$\vec{J}_f \propto \vec{E} \Rightarrow \boxed{\vec{J}_f = g\vec{E}}$$

g = "conductivity" E-field inside the conductor

(Ohm's law is only valid for free charges: bound charges do not contribute currents in steady fields)

Ohm's local law is equivalent to saying that the current density and thus the drift velocity are proportional to the electric field and thus the electric force $\vec{F} = q\vec{E}$. In fact $\vec{J}_f \propto \vec{v}_{drift} \propto \vec{F}$

How is it that the velocity (not the acceleration) is proportional to the force ???

We know from mechanics that for a single particle (charge), the (steady-state) velocity is proportional to the external force if a viscous friction force is present: the friction force grows with the speed until it is equal to the applied external force (example: free fall of an object through air). If the drag force is written as $\vec{F}_{drag} = -b\vec{v}$, then the steady state drift velocity is $v_{drift} = eE/b$, so that $J = \rho v_{drift} = Nev_{drift} = \left(Ne^2/b\right)E$, hence $g = Ne^2/b$

The viscous drag in a metal is due to the effect of collisions ; you'll see in the solid state physics course that these collisions are not between electrons and nuclei/ions, as in a classical picture, but are instead scattering processes off impurities, defects, and lattice vibrations (*phonons*). It is this scattering that is responsible for the resistance *R* of a metal.

Note that Ohm's law does not hold for free charges in vacuum: Newton's law predict in such case that it is the acceleration, not the velocity, that is proportional to the force per unit charge, hence the electric field: $\dot{\vec{v}} = \vec{a} \propto \vec{f} = \vec{E}$. The resultant non-Ohmic equation $d\vec{J}_f/dt \propto \vec{E}$ holds in superconductors, for which the resistance *R* is zero (!) .

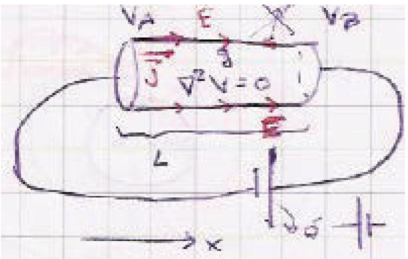
Resistance, global Ohm's law, Joule's law

For a homogeneous medium Ohm's law is $\vec{J}_f = g\vec{E}$, where the conductivity *g* is a constant. In steady conditions we have $\vec{J}_b = 0$ and by charge conservation: $\vec{\nabla}\cdot\vec{J}_f = \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} = 0$. Hence we get:

$$g(\vec{\nabla}\cdot\vec{E}) = 0 \Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{E} = \dfrac{\rho_{\text{tot}}}{\varepsilon_0} = 0 \\ \nabla^2 V = 0 \end{cases}$$

Therefore inside the bulk of a homogeneous conductor:
- → the net charge is zero even in the presence of a current
- → Laplace's law holds

Simple example: cylindrical conductor of length ℓ and conductivity *g*



Since in steady condition there is no build-up at the surface, it is: $\vec{J}_f \cdot \hat{n} = 0$ (boundary condition for **J**), which implies $\vec{E}\cdot\hat{n} = 0$
It is easy to find a solution to Laplace's equation that satisfies the boundary condition that the normal component of **E** is zero at the cylinder's lateral surface; namely: $V(x) = V_A + (V_B - V_A)\frac{x}{\ell}$

The uniqueness theorem tells us that this is the solution.
Thus we find that **J** and **E** are uniform inside the cylinder: $E = \frac{(V_A - V_B)}{\ell} \Rightarrow J_f = gE = g\frac{(V_A - V_B)}{\ell}$
So:

$$I = J_f A = g\frac{(V_A - V_B)}{\ell} \Rightarrow \boxed{\Delta V = IR} \quad \text{where} \quad \boxed{R = \frac{1}{g}\frac{\ell}{A} = \eta\frac{\ell}{A}}$$

(global) Ohm's law

is the *resistance* (*η* = 1/g = *resistivity*)

The power dissipated in the metal (due to Joule heating) is $\wp = I\,\Delta V = (\Delta V)^2/R = I^2 R$ ***Joule's law***

The resistance of the wire of a light bulb changes with T, being 100 Ω at room T and 1300 Ω at the operating temperature of 2500 °C. Why do light bulbs usually break when you turn them on?

Mobility; conductivity values for materials

Ohm's law states that the free current density is directly proportional to the macroscopic E-field: $\vec{J}_f \propto \vec{E}$. Since $\vec{J} = \rho\vec{v}_{drift} = nq\vec{v}_{drift}$, also the drift velocity is proportional to **E**: $\vec{v}_{drift} \propto \vec{E}$.
The proportionality coefficient is called mobility (μ) : $\vec{v}_{drift} = \mu\vec{E}$

$$\left.\begin{aligned} \vec{J}_f &= g\vec{E} \\ \vec{v}_{drift} &= \mu\vec{E} \\ \vec{J} &= ne\vec{v}_{drift} \end{aligned}\right\} \Rightarrow \boxed{g = nq\mu = \rho\mu}$$

R is measured in ohms (Ω, 1 Ω = 1 V/A)
g is measured in Ω^{-1} m^{-1}, or S/m
1 S (Siemens) = Ω^{-1}

Metals	*g* (S/m) at 20 °C
Silver	6.3×10^7
Copper	6×10^7
Gold	4.1×10^7
Chromium	3.8×10^7
Aluminium	3.5×10^7
Calcium	3×10^7
Tungsten	1.8×10^7
Zinc	1.7×10^7
Brass (65.8 Cu 34.2 Zn)	1.6×10^7
Nickel	1.4×10^7
Lithium	1.1×10^7
Iron	1×10^7
Platinum	9.4×10^6
Tin	9.2×10^6
Carbon steel (1010)	6.99×10^6
Lead	4.55×10^6
Titanium	2.38×10^6
Stainless steel	1.45×10^6
Mercury	1.02×10^6

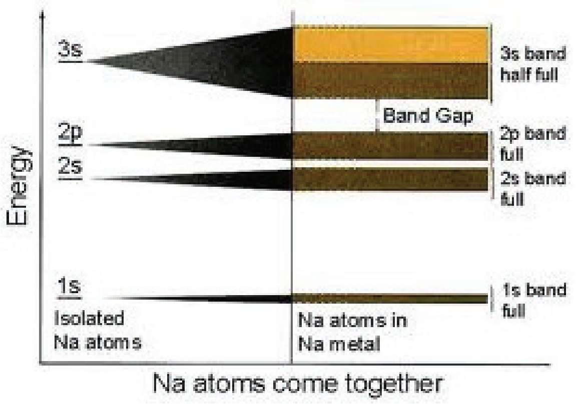
Semiconductors, half-metals, and liquid electrolytes	*g* (S/m) at 20 °C
Amorphous carbon	1.25×10^3 to 2×10^3
Graphite	2×10^5 to 3×10^5 //basal plane 3.3×10^2 ⊥basal plane
Silicon	1.56×10^{-3}
PEDOT:PSS	1×10^1 to 1×10^3
GaAs	5×10^{-8} to 10^3
Germanium	2.17
Sea water	4.8
Drinking water	5×10^{-4} to 5×10^{-2}
Deionized water	5.5×10^{-6}

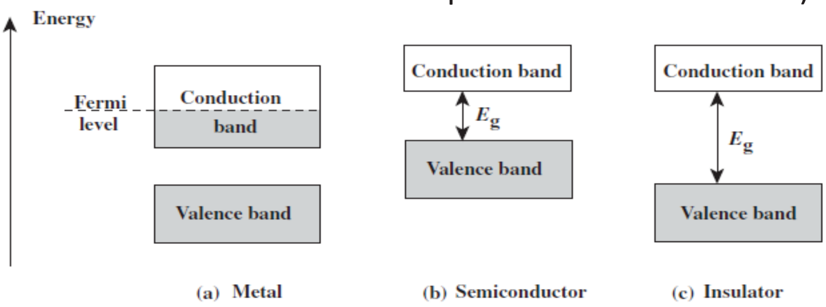
Insulators	*g* (S/m) at 20 °C
Diamond	$\sim10^{-13}$
Glass	10^{-11} to 10^{-15}
Hard rubber	10^{-14}
Wood (oven dry)	10^{-16} to 10^{-14}
Wood (damp)	10^{-4} to 10^{-3}
Sulfur	10^{-16}
Air	3×10^{-15} to 8×10^{-15}
Fused quartz	1.3×10^{-18}
PET	10^{-21}
Teflon	10^{-25} to 10^{-23}

Conductivity & classification of materials

- Electronic conductivity →
- *metals* $g \sim 10^{+6} - 10^{+7}$ S/m
- *semiconductors & half-metals* $g \sim 10^{-8} - 10^{+5}$ S/m
- *insulators* $g \sim 10^{-25} - 10^{-10}$ S/m

(g is strictly zero only for perfect insulators **at 0 K**)





(demonstration of existence of conduction electrons in metals: discovery of electric inertia (1913), from which e/m was measured

- Ionic conductivity → In solids only one type of ion is mobile, usually cations (H^+, Li^+, Na^+, K^+, Ni^{2+}, Ni^{4+},...) as they are smaller (in the same solid, mobility(Na^+) > mobility(K^+), etc). Instead, in liquid solutions all ions are mobile.

electrolytes
(ion migration)

Solid-state ionic conductivity is important:
1) for electrical insulation, as it worsens the insulating properties of a material; for example, in glass (SiO_2) insulation, the presence of impurities such as alkali oxides lowers the resistance, the presence of heavier-metal oxides such as BaO or PbO increases it
2) in applications such as batteries for portable devices, rechargeable batteries, or fuel cells

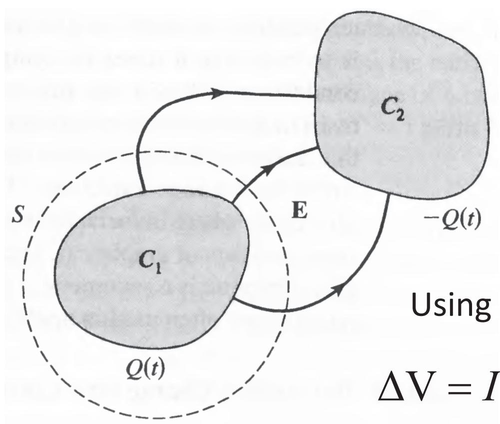
Homogeneous polarizable conductors

homogeneous medium: $\begin{cases} g(\vec{r}) = g = const \\ \varepsilon_r(\vec{r}) = \varepsilon_r = const \end{cases} \Rightarrow \begin{cases} \vec{J}_f(\vec{r}) = g\vec{E}(\vec{r}) \\ \vec{D}(\vec{r}) = \varepsilon_r\varepsilon_0\vec{E}(\vec{r}) \end{cases}$

The charge conservation equation and the condition of steady state together imply:

$$\left.\begin{aligned} \vec{J}_b = \frac{\partial \vec{P}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = 0 \end{aligned}\right\} \Rightarrow \vec{\nabla}\cdot\vec{J}_f = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} = 0$$

Since in the bulk of a homogeneous medium $\rho_{\text{tot}} = \dfrac{\rho_f}{\varepsilon_r}$, with **steady currents** $\boxed{\rho_f = \rho_{tot} = 0}$



General relationship between C and R between **2 conductors in a homogeneous medium :**

$$I_f = \oint_S \vec{J}_f \cdot d\vec{a} = g\oint_S \vec{E}\cdot d\vec{a} = \frac{g}{\varepsilon_r\varepsilon_0}\oint_S \vec{D}\cdot d\vec{a} = \frac{g}{\varepsilon_r\varepsilon_0} Q_{f\,\text{in}\,S}$$

Using the definitions of resistance and capacitance,

$\Delta V = I_f R$ and $C = \dfrac{Q_f}{\Delta V}$, we finally obtain : $\boxed{RC = \dfrac{\varepsilon_r\varepsilon_0}{g}}$

To do @home: show that the product RC has the dimensions of time (seconds), & demonstrate that in a RC circuit, the decay of the charge of the capacitor during discharge is ∝ exp(–t/RC)

Inhomogeneous conductors

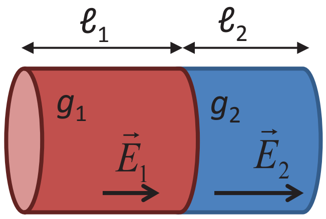
$$\begin{cases} \vec{J}(\vec{r}) = g(r)\vec{E}(\vec{r}) \\ \vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\left(g(\vec{r})\vec{E}(\vec{r})\right) = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} \neq 0 \ ! \end{cases}$$

→ Laplace's law doesn't hold for a inhomogeneous conductor

→ we can't solve as we did for a homogeneous conductor. Instead, we can use the continuity condition: in steady state the total flow of charge (current) through a cross-section of the conductor must equal the total flow through any other. It is found that, in general, **when a current runs through an inhomogeneous medium, the charge density is stationary but nonzero**

Example: Junction of two conductors of same cross section but with $g_1 \neq g_2$, carrying a current I:

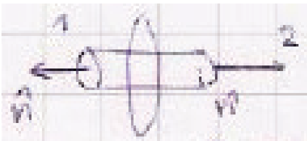
$$I_1 = I_2 \Rightarrow \vec{J}_1 = \vec{J}_2$$



$$\left.\begin{array}{l} \vec{J}_1 = g_1\vec{E}_1 \\ \vec{J}_2 = g_2\vec{E}_2 \\ \vec{J}_1 = \vec{J}_2 \end{array}\right\} \Rightarrow \vec{E}_2 = \frac{g_1}{g_2}\vec{E}_1$$

$$\Delta V = E_1\ell_1 + E_2\ell_2 = \frac{J_1\ell_1}{g_1} + \frac{J_2\ell_2}{g_2} = \frac{I\ell_1}{g_1 S} + \frac{I\ell_2}{g_2 S} \quad \left(\Rightarrow R = \frac{\Delta V}{I} = \frac{\ell_1}{g_1 S} + \frac{\ell_2}{g_2 S} = R_1 + R_2\right)$$

Boundary condition: $\vec{E}_1\cdot\hat{n} + \vec{E}_2\cdot\hat{n} = \frac{\sigma}{\varepsilon_0}$; hence $\vec{E}_2 \neq \vec{E}_1 \Rightarrow \sigma \neq 0$!!!



$$-E_1 + E_2 = \frac{\sigma}{\varepsilon_0} \rightarrow -\frac{J}{g_i} + \frac{J}{g_2} = \frac{\sigma}{\varepsilon_0} \Rightarrow \sigma = \varepsilon_0 J\left(\frac{1}{g_2} - \frac{1}{g_1}\right) = \varepsilon_0 \frac{I}{S_{\substack{\text{cross}\\\text{section}}}}\left(\frac{1}{g_2} - \frac{1}{g_1}\right)$$

Problems with non-polarizable metals

2 types of problems (the text of the problem will specify the value of $g(\mathbf{r})$) :

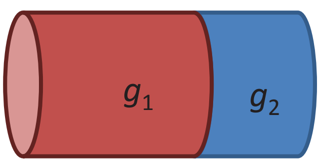
(a) highly symmetric, with the separation between media running orthogonal to E, or else spherical or cylindrical;

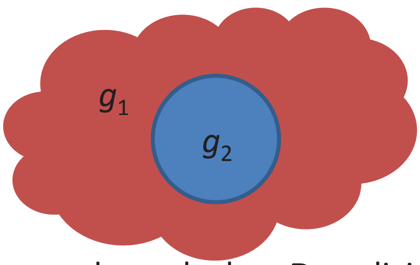
(b) Less symmetric, with the separation between media running parallel to E

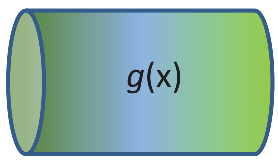
Strategy for case (a): guess the symmetry of **J**, hence impose the boundary condition for **J** (the normal component of **J** is constant, or equivalently, total (free) current I must be the same through any cross-section perpendicular to **J**. Calculate **E** from Ohm's local law (it will be different in different conductors and will be a function of position in non-homogeneous conductors). If possible, integrate **E** to find V and thus obtain the resistance as R = V/I .

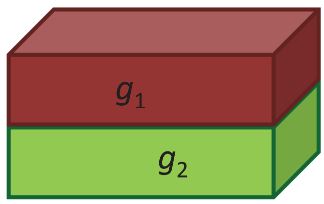
If required, use **E** to find $\rho_{tot} = \rho_f$ (for nonhomogeneous conductors) and $\sigma_{tot} = \sigma_f$ at boundaries.

Strategy for case (b): study the boundary conditions for **E** to guess its direction and intensity (the component parallel to a separation will be continuous); check that you get a unique value of V integrating over different paths. Use Ohm's law to calculate **J** in the various conductors, hence calculate the total current I to obtain the resistance as R = V/I. Use **E** find ρ_{tot} and σ_{tot} .









In either case, there is no need to calculate **D** explicitly, since in a metal **P** = 0 (so $\rho_b = \sigma_b = 0$) and therefore **D** = ε_0 **E** (that is, in a metal $\varepsilon_r = 1$); the discontinuities of **D** are exactly the same as those of **E**, since $\rho_{tot} = \rho_f$ and $\sigma_{tot} = \sigma_f$

Problems with non-ideal dielectrics

IMPORTANT CONSTRAINTS to be fulfilled (for **E** and **J**):

$$(\vec{\nabla}\times\vec{E}=0 \Rightarrow)\quad E_{t1}=E_{t2} \qquad I=\int \vec{J}\cdot d\vec{a}=const$$

$$V(\vec{r})=-\int \vec{E}(\vec{r})\cdot d\vec{\ell} \qquad \vec{\nabla}\cdot\vec{J}_f=0$$

2 types of problems (the text of the problem will specify the value of g(**r**) and of ε_r(**r**)) :
(a) highly symmetric, with the separation between media running orthogonal to E
(b) Less symmetric, with the separation between media running parallel to E

Strategy for case (a): guess the symmetry of **J** and express it in terms of the total (free) current I Calculate **E** from Ohm's local law, integrate it to find V and thus obtain the resistance R. Calculate **D** and **P** using the constitutive equation. Use **E**, **D** and **P** to find ρ_{tot} , ρ_f and ρ_b (for nonhomogeneous media) and σ_{tot} , σ_f and σ_b (at boundaries).

Strategy for case (b): study the boundary conditions for E to guess the symmetry of E (which for example for a capacitor will be the same as that of the empty capacitor); from **E** find V and use the constitutive equation to find **D** and **P**. As before, find all charge densities.

In the case of a capacitor, from the surface free-charge density σ_f on the plates, calculated using the boundary condition for **D**, we can calculate the total (free) charge on the plate, Q_f. Hence using the value of V found previously we can get the capacitance as $C = Q_f/V$
Note however that a capacitor with a non-ideal dielectric is a leaky capacitor, since the charge on the plates is constantly diminishing due to the current flowing through the dielectric. Moreover, if the conductivity of the dielectric is not homogeneous, free charge accumulates in the dielectric, so that one cannot even define a capacitance, as there is free charge outside the two conductors (though one can still define an "effective" capacitance, see problem classes)

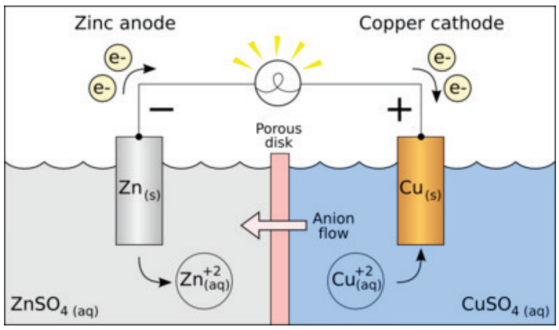
Generation of steady currents

Working principle of a Voltaic cell, battery, or fuel cell: conversion of "chemical" energy (stored in the materials/fuel) into flowing electrical energy (current). The "chemical" energy is actually the difference in electrostatic energy of 2 different microscopic (atomic) electronic configurations

$$U_{chemical}(q+dq)-U_{chemical}(q)=\mathcal{E}\,dq$$

$$\Rightarrow \boxed{\mathcal{E}_{elec}=\frac{dU_{chemical}}{dq}}$$

dq = charge dissociating at anode/ associating at cathode

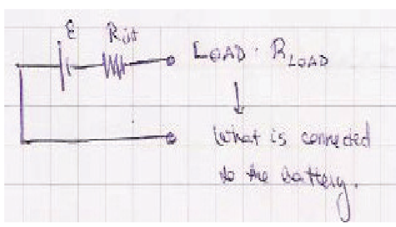


When integrating over a closed loop, $\oint_{LOOP}\vec{E}\cdot d\vec{\ell}=\mathcal{E}_{elec}=0$. However, if a battery is connected, the "closed loop" actually goes through the battery. One then separates the integral into two parts, one over the battery and one over the rest of the circuit :

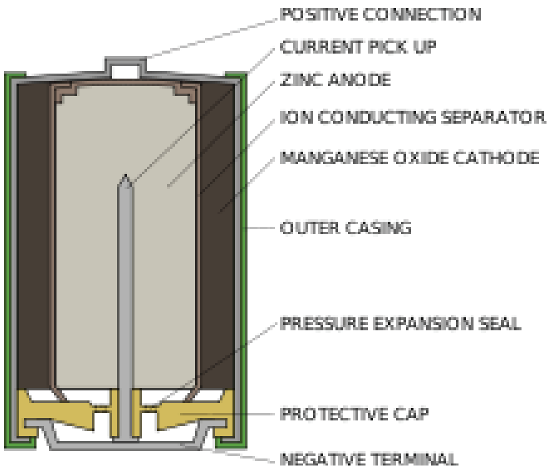
$$\oint_{loop}\vec{E}\cdot d\vec{\ell}=\mathcal{E}_{battery}+\int_{\text{rest of circuit}}\vec{E}\cdot d\vec{\ell}=\mathcal{E}_{battery}+(-\Delta V)=0 \quad\Rightarrow\quad \boxed{\mathcal{E}_{battery}=\Delta V}$$

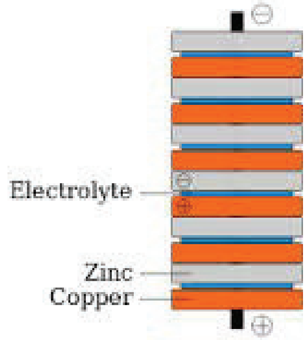
When connected, charges (counterions) flow inside the battery: hence it behaves as if it had an "internal" resistance : →

$$\boxed{\mathcal{E}_{elec}=(R_{int}+R_{load})I}$$



*Alcaline batteries





the first battery was invented by A. Volta (voltaic pile)

Anode (Zn)	Electrolyte: KOH (*aq*)	Cathode (MnO_2)
$Zn + 2OH^- \rightarrow ZnO_2^{2-} + 2H^+ + 2e^-$	$\downarrow K_2ZnO_2$ *precipitates*	$2MnO_2 + H_2O + 2e^- \rightarrow$ $\rightarrow Mn_2O_3 + 2OH^-$

overall reaction:

$$Zn(s) + 2MnO_2(s) + 2KOH(aq) \rightarrow K_2ZnO_2(s) + Mn_2O_3(s) + H_2O$$

*Rechargeable batteries

If the electrochemical reactions takig place during discharge of a battery can be reversed by changing the sense in which the current flows, then the battery is rechargeable. Examples:

Car batteries

Anode (Pb)	Electrolyte (H_2SO_4 (*aq*))	Cathode (PbO_2)
$Pb + SO_4^{2-} \rightarrow PbSO_4 + 2e^-$	overall reaction:	$PbO_2 + 4H^+ + SO_4^{2-} + 2e^- \rightarrow$ $\rightarrow PbSO_4 + 2H_2O$

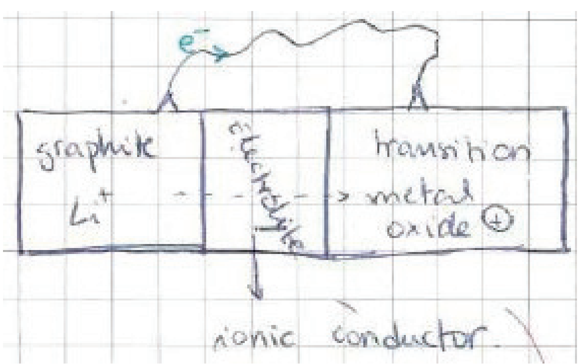
discharge →

$$Pb(s) + PbO_2(s) + 2H_2SO_4(aq) \leftrightarrow 2PbSO_4(s) + 2H_2O$$

← *(re)charge*

You should avoid leaving the battery uncharged for long times, otherwise $PbSO_4$ crystals so large will form that the reverse reaction will no longer be able to take place

Lithium-ion solid-state batteries

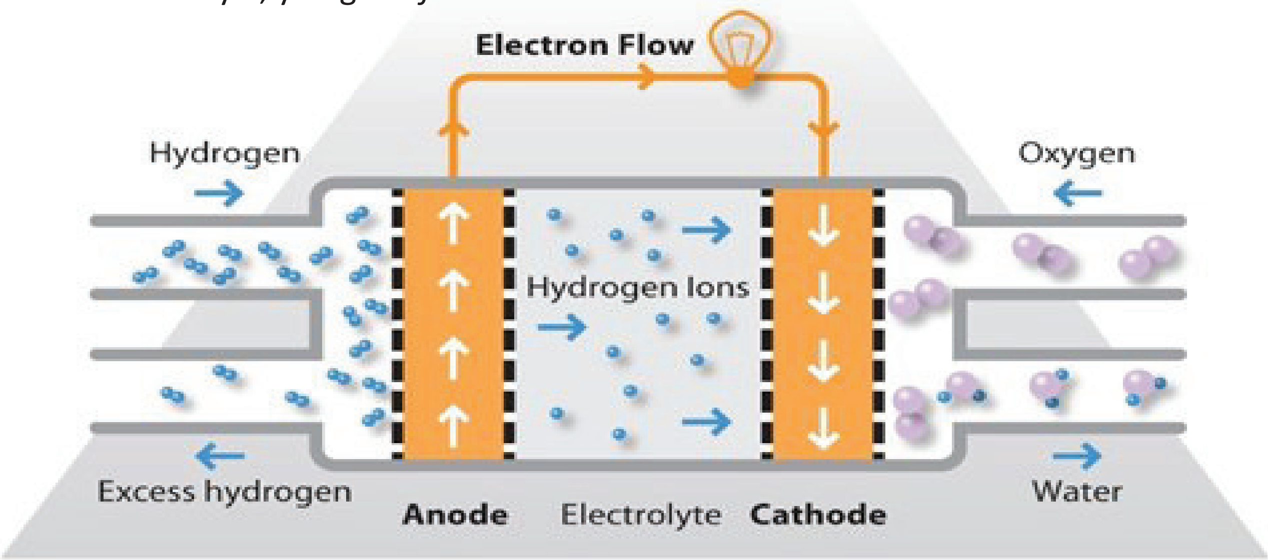


In these batteries the electrolyte is a solid instead of a liquid solution; the solid electrolyte is an ionic conductor, permeable to Li ions but not to electrons It is preferred for safety reasons to have batteries without liquids in portable applications such as mobiles, laptops, gps systems, etc.

Each time a Li^+ ion moves, there's a variation of chemical energy, due to the fact that one material has more affinity for Li^+ ions than the other

*Fuel cells

If instead of using solid electrodes to providing the source of ionic species, you use a gas (H_2, O_2) or a liquid (e.g. methanol, CH_4O) and favor decomposition of such gas/liquid at the electrodes by means of a catalyst, you get a *fuel cell*:



Anode (carbon paper with Pt particles)

$$\mathrm{H_2 \rightarrow 2H^+ + 2}e^-$$

Electrolyte: silicon carbide (SiC) saturated with H_3PO_4

Cathode (same as anode)

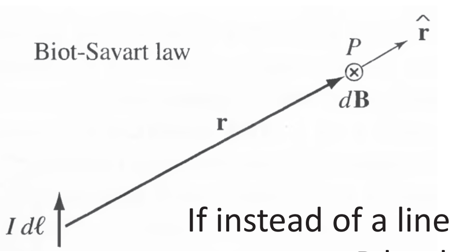
$$\frac{1}{2}\mathrm{O_2 + 2}e^- \rightarrow \mathrm{O^{2-}}$$

Electrolyte-cathode interface

$$\mathrm{2H^+ + O^{2-} \rightarrow H_2O}$$

overall reaction: $\frac{1}{2}\mathrm{O_2 + H_2 \rightarrow H_2O}$

B-field generated by currents



Biot-Savart's law

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}\times\hat{r}}{r^2} \Rightarrow \boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int Id\vec{\ell}\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}}$$

If instead of a line current I we have a volume (**J**) or a surface **(K)** current density, we can get B by the replacement: $\int_{line}()Id\ell \approx \int_{surface}()Kda \approx \int_{volume}()Jd\tau$

vector potential A for B-field generated by currents: $\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r})$ PROOF:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau'\,\vec{J}(\vec{r})\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{\mu_0}{4\pi}\int d\tau'\,\vec{J}(\vec{r})\times\left(-\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}\right) =$$

$$= \frac{\mu_0}{4\pi}\int d\tau'\,\vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\times\vec{J}(\vec{r}') = \frac{\mu_0}{4\pi}\int d\tau'\,\vec{\nabla}\times\left(\frac{\vec{J}(\vec{r}')}{\vec{r}-\vec{r}'}\right) \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla}\times\underbrace{\left[\frac{\mu_0}{4\pi}\int d\tau'\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right]}_{\vec{A}(\vec{r})}$$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau'\frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}}$$

(notice that the vector potential **A** of a current element $Id\ell$ = ***J*** dτ is always parallel to the current)

If there is no build-up of charge ($\partial\rho/\partial t = 0$), then by charge conservation we have: $\vec{\nabla}\cdot\vec{J} = 0$.
It can be shown (see exercise 0-17) that if ***J*** is limited to a finite region of space:

$$\vec{\nabla}\cdot\vec{J} = 0 \quad\Rightarrow\quad \vec{\nabla}\cdot\vec{A} = 0 \quad \text{(true in magnetostatics)}$$

Ampère's law

$$\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$ **Gauss's law for B**

Moreover, in magnetostatics:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{A}}_{\vec{\nabla} \cdot \vec{A} = 0}) - \nabla^2 \vec{A} = -\nabla^2 \left[\frac{\mu_0}{4\pi} \int d^3 r' \frac{\vec{J}(\vec{r}')}{\vec{r} - \vec{r}'}\right]$$

$$= \frac{\mu_0}{4\pi} \int d^3 r' \vec{J}(\vec{r}') \left[-\nabla^2 \frac{1}{|\vec{r} - \vec{r}'|}\right] = \frac{\mu_0}{4\pi} \int d^3 r' \vec{J}(\vec{r}') 4\pi \delta^3 (\vec{r} - \vec{r}') = \mu_0 \vec{J}(\vec{r})$$

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Leftrightarrow \oint \vec{B} \cdot d\vec{\ell} = \mu_0 I}$$

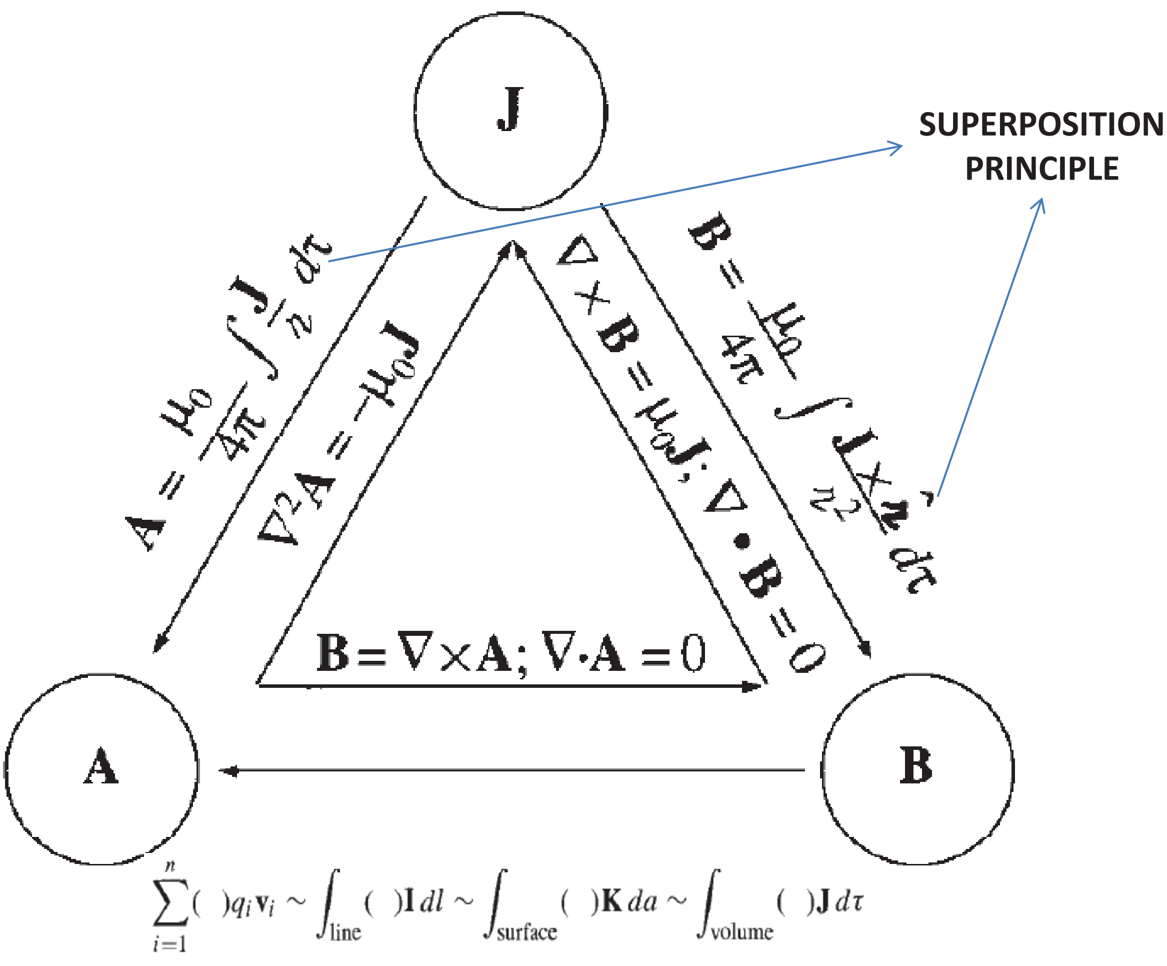
Ampere's law (differential and integral forms)

(Stokes' integral theorem)

Note that Ampère's law can only be true for steady currents for which there is no accumulation of charge (local charge density = constant), for otherwise it would violate charge conservation!!

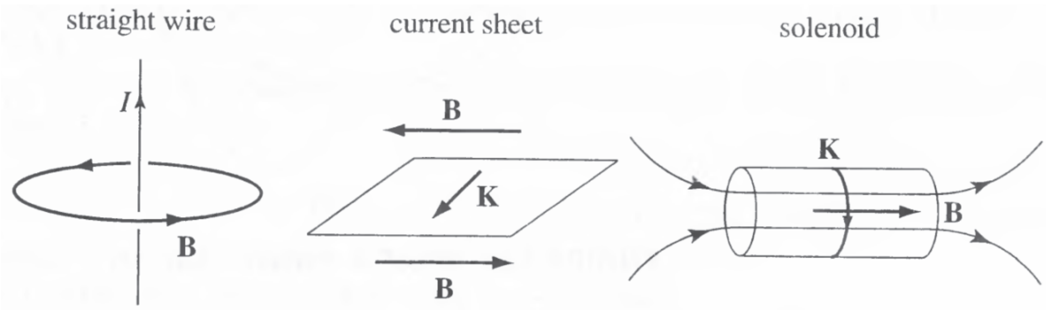
Simple example vector potential: : vector potential of uniform B-field (see problem 4-22 and Txtbk p. 281)

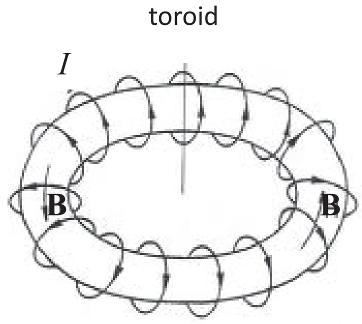
Graphical summary of magnetostatics



Biot Savart's law vs Ampere's law

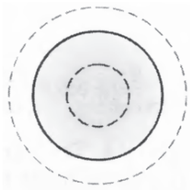
Use: Ampere's in symmetric cases, Biot-Savart's law in other cases



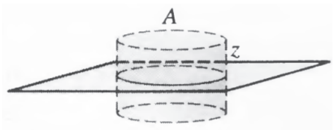


Compare with useful symmetries with Gauss's law :

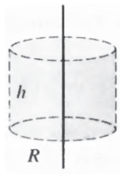
spherical



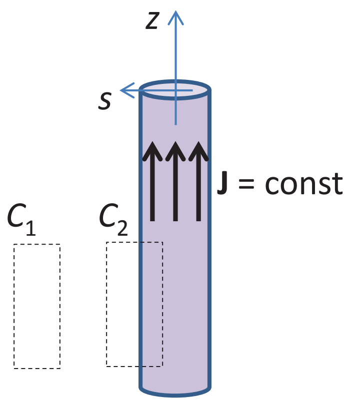
planar



cylindrical



Another use of Ampere's law: if $\vec{B} = axy\hat{x} + by^2\hat{y}$, what $\vec{J}$ generates this field?



Example 1: infinite wire

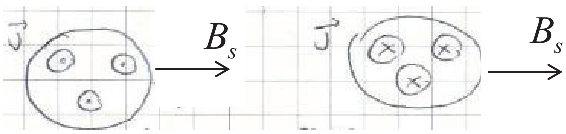
We assume the wire (or radius *a*) is very long (infinite) and carries a uniform (free) current density

1) By symmetry, $\vec{B}$ cannot depend on z nor φ $\Rightarrow$ $\vec{B} = \vec{B}(s)$

2) By Biot-Savart's law, when $s \to \infty$, H and $B \to 0$

Cylindrical coordinates → $\vec{B} = \left(B_z, B_s, B_\varphi\right)$

3) wire seen from above and below:



But changing the sign of $\vec{J}_f$ must reverse B ! $\Rightarrow B_s = 0$
(again by Biot-Savart's law)

4) Both outside and inside, $B_z = 0$ (Ampère's law along C_1 or C_2 , with left side → ∞)

5) $\Rightarrow$ only one component $B_\varphi \neq 0$, it depends only on the radial coordinate s

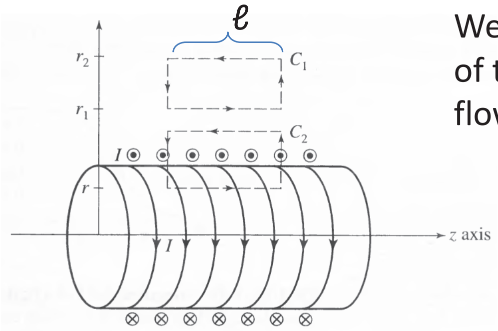
→ actual calculation of B !!!

$\vec{B} = \left(0, 0, B_\varphi(s)\right)$

Inside: $\int \vec{B} \cdot d\vec{\ell} = \mu_0 \overbrace{J\pi s^2}^{I(s)} \Rightarrow B_\varphi = \mu_0 \dfrac{Js}{2}$

Outside: $\int \vec{B} \cdot d\vec{\ell} = \mu_0 I \Rightarrow B_\varphi = \mu_0 \dfrac{I}{2\pi a}$

Example 2: infinite solenoid



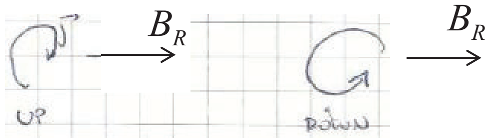
We assume the solenoid is very long (infinite) and thin (thickness of the wire is negligible), and made of flat coils → the current flow along a plane parallel to the *xy* plane is basically zero

1) By symmetry, $\vec{B}$ cannot depend on z nor φ $\Rightarrow \vec{B} = \vec{B}(s)$

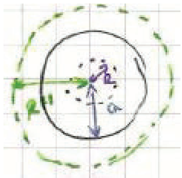
2) By Biot-Savart's law, when $s \to \infty$, H and $B \to 0$

Cylindrical coordinates → $\vec{B} = \left(B_z, B_s, B_\varphi\right)$

3) solenoid seen from above and below:



But changing the sign of $\vec{J}_f$ must reverse B ! $\Rightarrow$ $B_s = 0$
(again by Biot-Savart's law)



4) Applying Ampère's law to thes circles shown, we get that both outside and inside $B_\varphi = 0$ (since the current through a circle is basically zero)

→ actual calculation of B !!!

5) ONLY outside $B_z = 0$ (Ampère's law along C_1, with top side → ∞)

Inside: B ℓ = *N* I (Ampère's law along C_2)

$$\Rightarrow B = \mu_0 \left(\frac{N}{\ell}\right) I = \mu_0 n I$$, where *n* = number of turns per unit length

B- & H- fields of currents & magnetized media

B-field

- with only currents in vacuum or non-magnetic media: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$; $\vec{\nabla} \cdot \vec{B} = 0$
- with only magnetized media: $\vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \times \vec{M}$; $\vec{\nabla} \cdot \vec{B} = 0$

→ COMBINE THEM !! (SUPERPOSITION PRINCIPLE)

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{\nabla} \times \vec{M} \Rightarrow \vec{\nabla} \times \vec{B} - \mu_0 \vec{\nabla} \times \vec{M} = \mu_0 \vec{J} \Rightarrow \vec{\nabla} \times \underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)}_{\vec{H}} = \vec{J}$$

$\Rightarrow$ Maxwell's equations for magnetostatics

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{H} = \vec{J}_f \end{cases}$$

→ Ampère's law for **H**

$$\oint \vec{H} \cdot d\vec{\ell} = I$$

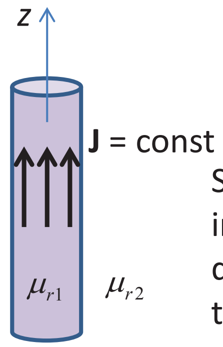
If the magnetized medium is lineal (hence not a magnet), we also have $\begin{cases} \vec{B} = \mu_r \mu_0 \vec{H} \\ \vec{M} = \chi_m \vec{H} \end{cases}$

$$\oint \vec{H} \cdot d\vec{\ell} = I \Rightarrow \left[\vec{H}\right] = \left[\vec{M}\right] = \left[\sigma_m\right] = \text{A/m} \quad \text{and} \quad \left[\rho_m\right] = \text{A/m}^2$$

Problems with currents & linear magnetic media

Ex. 1) & 2) with linear media:

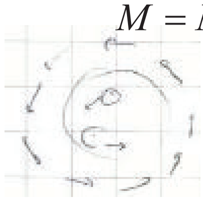
1)



We found that the only component of **B** and therefore of the **H** field created by the current is the azimuthal one: only $H_\varphi \neq 0$

The field $\vec{H}_J = H_\varphi \hat{\varphi}$ acts as stimulus field that magnetizes the linear material of the wire and of the medium surrounding it.

$\vec{M} = M_\varphi \hat{\varphi}$



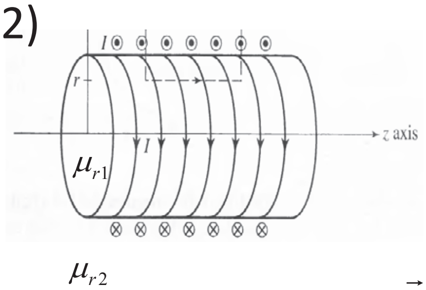
Since **H** is azimuthal, we guess that **M** will be too, both inside and outside ; in such case, the magnetic pole density is everywhere zero, since M is perpendicular to the boundary and it is moreover divergenceless:

$$\begin{cases} \rho_m = -\vec{\nabla}\cdot\vec{M} = 0 \\ \sigma_m = \vec{M}\cdot\hat{n} = 0 \end{cases}$$

Therefore the total (macroscopic) H-field is simply the one produced by the current, since there are no poles; hence we may write:

$$\vec{M}_{1,2} = \chi_{m1,2}\vec{H}_{macro} = \chi_{m1,2}\vec{H}_J\ , \qquad \vec{B}_{1,2} = \mu_{r1,2}\mu_0\vec{H}_{macro} = \mu_{r1,2}\mu_0\vec{H}_J$$

2)



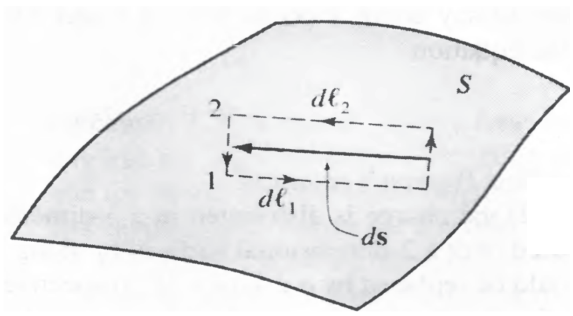
We found that **H** is parallel to *z* and nonzero only inside. We guess that **M** will be axial inside and zero outside. The corresponding magnetic pole density is limited to the flat surfaces of the solenoid, and in principle create a response field $\mathbf{H}_{resp}$; however, in the limit of an infinitely long solenoid the poles are infinitely far away, so $\mathbf{H}_{resp} \approx 0$, and again we find:

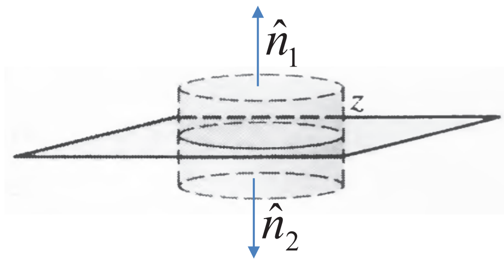
$$\vec{M}_{1,2} = \chi_{m1,2}\vec{H}_{macro} = \chi_{m1,2}\vec{H}_J\ , \qquad \vec{B}_{1,2} = \mu_{r1,2}\mu_0\vec{H}_{macro} = \mu_{r1,2}\mu_0\vec{H}_J$$

Boundary conditions for magnetostatics

Maxwell's equations for magnetostatics with currents and magnetic media:

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{H} = \vec{J}_f \end{cases}$$





$$\vec{\nabla}\cdot\vec{B} = 0 \;\Rightarrow$$

$$\Phi(\vec{B}) = \vec{B}_1\cdot\hat{n}_1 A + \vec{B}_2\cdot\hat{n}_2 A = 0$$

$\Rightarrow$ The normal component of **B** is conserved:

$$\vec{B}_1\cdot\hat{n}_1 + \vec{B}_2\cdot\hat{n}_2 = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J}_f \;\Rightarrow\; \oint \vec{H}\cdot d\vec{\ell} = I$$

With $d\vec{x}$ in the surface direction orthogonal to $d\vec{\ell}$,

one has: $I d\vec{x} = \vec{K} da = \vec{K}|d\ell| dx$. Hence:

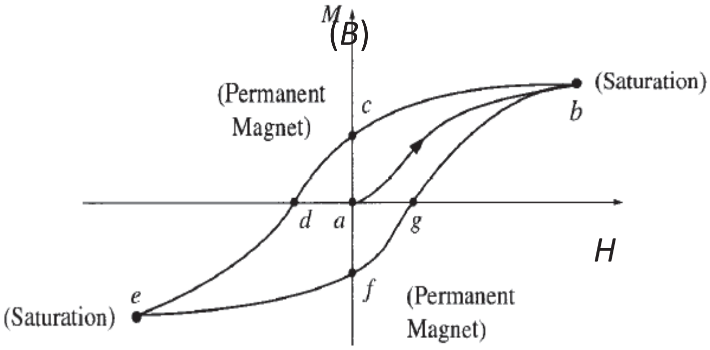
$$H_{t2}d\ell_2 + H_{t1}d\ell_1 = (H_{t2} - H_{t1})|d\ell| = I = K|d\ell|$$

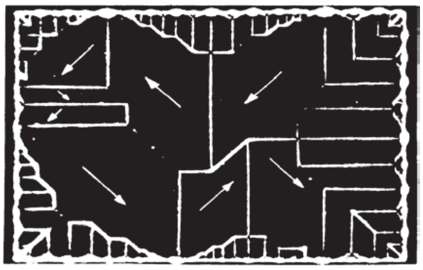
$$\Rightarrow \vec{H}_1\times\hat{n}_1 + \vec{H}_2\times\hat{n}_2 = \vec{K}$$

(or also: $\vec{H}_{t,1} - \vec{H}_{t,2} = \vec{K}\times\hat{n}$ with $\vec{H}_t = \vec{H} - \vec{H}\cdot\hat{n}$)

If the surface current density $\vec{K}$ is zero, which is the most common case in magnetostatics, the tangential component of **H** is conserved. Note however that $\vec{K} \neq 0$ for superconductors also under magnetostatic conditions, as we will see later in this topic (with time-varying fields, $\vec{K} \neq 0$ for electromagnetic waves impinging on metals; e.g. in the reflection of light by metallic mirrors)

Ferromagnetic materials in applied H

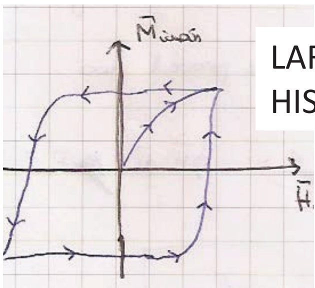




Ferromagnetic domains

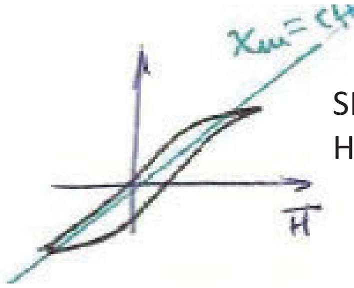
2 types of ferromagnetic materials:

- HARD ferromagnets :
hard to change their magnetization
→ used for permanent magnets)



LARGE HISTERESIS

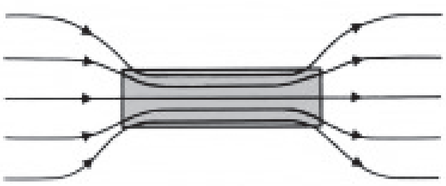
- SOFT ferromagnets :
easy to change their magnetization
→ used for shielding, electromagnets, ...



SMALL HISTERESIS

Applications of SOFT ferromagnetic materials

Example: soft ferromagnetic bar



Outside: $\vec{B}_{out} = \mu_0 \vec{H}$

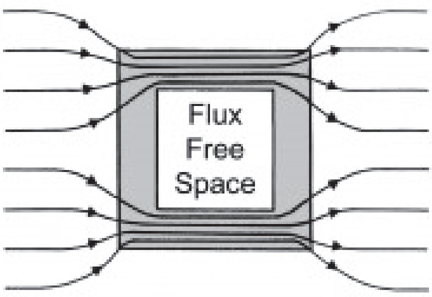
Inside: $\vec{B}_{in} = \mu_r \mu_0 \vec{H}$

$$\frac{\vec{B}_{in}}{\vec{B}_{out}} = \mu_r = 1000 \div 100000 \text{ !!!}$$

Concentration of the magnetic flux:

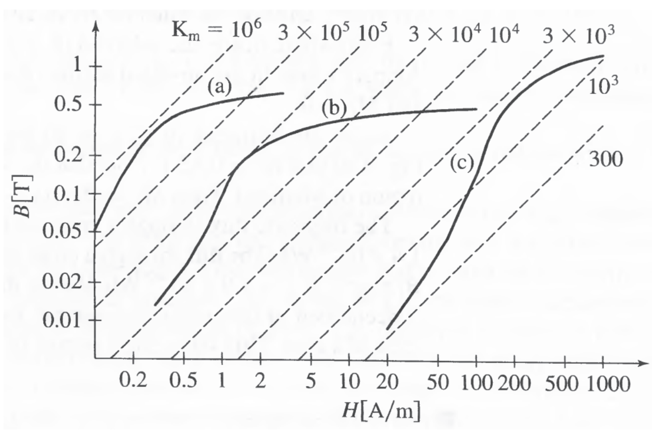
"AMPLIFICATION" of **B**

→ application: magnetic shield (see Pr. **3-14**)



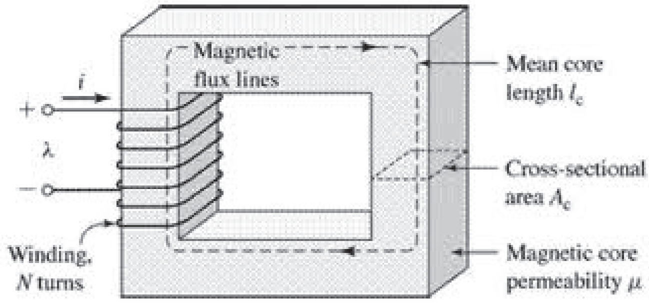
→ other applications:
magnetic circuits & electro-mechanical energy conversion (generators, motors, electromagnets, relays, transformers) → see topic 5

magnetization curves (B - H) for supermalloy (a), mumetal (b), soft iron (c)



magnetic circuits: zero-leakage approximation

Magnetic circuit: piece of "linear" ferromagnet that guides and confines in its interior the lines of B of a permanent magnet or coil with a current



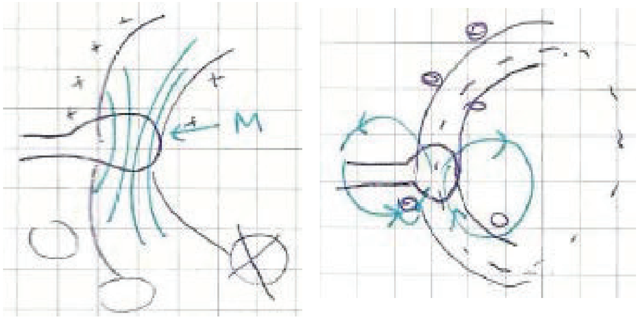
FIELDS in a MAGNETIC CIRCUIT:

(1) $\vec{B} \parallel \vec{M} \parallel \vec{H}$ everywhere outside permanent magnets (linear constitutive equation)

(2) $\Phi_B = \text{constant}$ (this is called zero-leakage approximation)

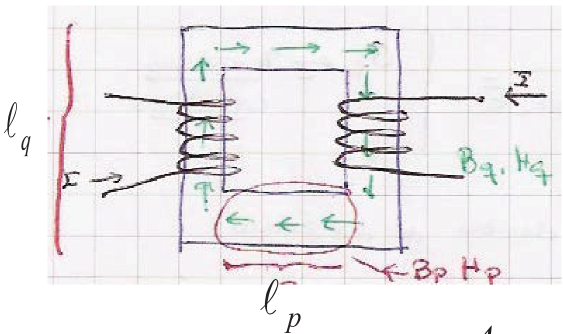
(3) $\oint \vec{H} \cdot d\vec{\ell} = NI$ (Ampère's law for **H** ; this equation is easy to apply only if (2) holds)

zero flux leakage:



The confinement of the magnetic field inside the material is due to the formation of localized magnetic "poles" in the region next to the coil. This is energetically more favorable than to have M everywhere parallel to the field lines generated by the coil alone, since it costs energy to form poles (domain boundaries). Note the analogy with electric circuits, where free electric charges bend the lines of E so that they remain inside the conducting wire

Hopkinson's law



$$\oint \vec{H} \cdot d\vec{\ell} = NI + NI = 2NI$$

$$H_q 2\ell_q + H_p 2\ell_p = 2NI$$

$$\Rightarrow H_q \ell_q + H_p \ell_p = NI = \frac{B_q \ell_q}{\mu_r \mu_0} + \frac{B_p \ell_p}{\mu_r \mu_0}$$

$$\Phi = B_p S_p = B_q S_q \Rightarrow B_p = B_q \frac{A_q}{A_p} \longrightarrow \Phi \underbrace{\frac{1}{A_q}\frac{\ell_q}{\mu_0 \mu_r}}_{\Re_q} + \Phi \underbrace{\frac{1}{A_p}\frac{\ell_p}{\mu_0 \mu_r}}_{\Re_p} = NI$$

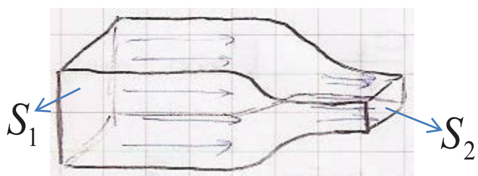
$\Rightarrow$ Hopkinson's law:

$$NI = (\Re_q + \Re_p)\Phi = \Re\Phi$$

$\mathscr{M} = N_1 I_1 + N_2 I_2$ is the MAGNETOMOTIVE FORCE (m.m.f.)

$\Re$ is the RELUCTANCE of the magnetic circuit (or part of it)

The same formula holds if a part of the circuit has different cross-section S, since in the zero leakage approximation: →

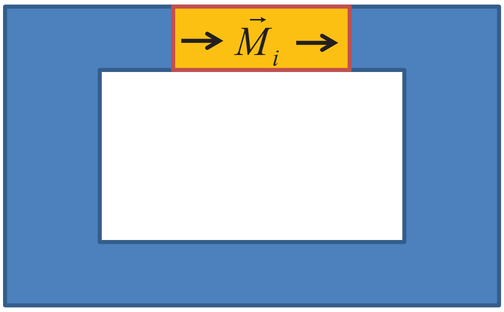


$$\Phi_1 = \Phi_2 \Rightarrow B_1 S_1 = B_2 S_2$$

Notice the similarity between Hopkinson's law and Kirchhoff's law :

$$\mathscr{M} = \Re\Phi \;, \quad \Re = \frac{1}{\mu_0 \mu_r}\frac{\ell}{A} \qquad\qquad \mathscr{E} = RI \;, \quad R = \frac{1}{g}\frac{\ell}{A}$$

Magnetic circuits with a permanent magnet



Consider a magnetic circuit consisting of a permanent magnet of magnetization M_i linked with a linear magnetic material. Call ℓ_i the length of the bar magnet and ℓ_c the (average) length of the rest of the circuit. Ampère's law is, in the absence of coils:

$$\oint \vec{H} \cdot d\vec{\ell} = NI = 0 = H_i \ell_i + H_c \ell_c$$

The line integral of **H** can be zero because **H** is opposite to **M** and **B** inside a permanent magnet, that is, because $H_i < 0$. If the cross section of the circuit is uniform, then in the zero-leakage approximation B is everywhere constant in amplitude.

In the linear medium we have $H_c = \dfrac{B}{\mu_r \mu_0}$; inside the magnet we have $H_i = \dfrac{B}{\mu_0} - M_i$

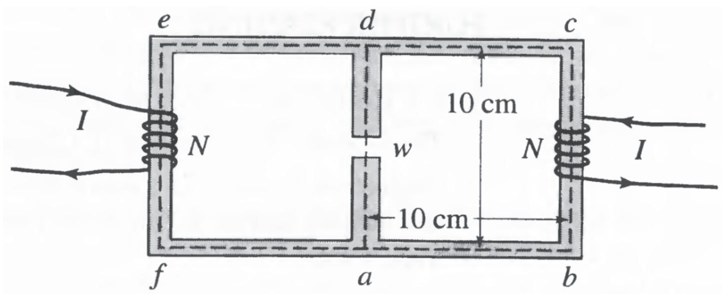
Combining these equations gives:

$$\left(\frac{B}{\mu_0} - M_i\right)\ell_i + \frac{B}{\mu_r \mu_0}\ell_c = 0 \quad \Rightarrow \quad M_i \ell_i = \underbrace{\left(\frac{1}{\mu_0}\frac{\ell_i}{A_i} + \frac{1}{\mu_r \mu_0}\frac{\ell_c}{A_c}\right)}_{\Re = \Re_1 + \Re_2}\Phi$$

→ Hopkinsons'law holds also for a magnetic circuit containing a permanent magnet, provided:
1) we include in the magnetomotance the term $M_i \ell_i$,
2) For the reluctance, we treat the region occupied by the magnet as if it were empty

Application: electromagnet design

A certain electromagnet consists of an iron yoke of relative permeability $\mu_r = 2000$ wound with 2 coils each with $N = 500$, with a gap of width $w = 1$ cm. What current I is needed to get a field of 1 T in the gap? Calculate the magnetic pole density at the gap



The B-field (H-field) along *ad* is B'_F (H'_F) in the iron and B_g (H_g) in the gap, elsewhere in the iron its value is B_F (H_F). Since the normal component of **B** is continuous across the gap surface, $B'_F = B_g$. Also, by symmetry the B-field in the iron along *ad* should be twice as large as that in other parts of the yoke: $B'_F = 2B_F$. Apply Ampere's law to the path *abcda* :

$$\oint_{abcda} \vec{H} \cdot d\vec{\ell} = 0.3H_F + (0.1 - 0.01)H'_F + 0.01H_g = 500I$$

In the gap we have $B_g = 1\,\mathrm{T} \Rightarrow H_g = \dfrac{B_g}{\mu_0} \approx 8 \cdot 10^5$ A/m . Since $B'_F = B_g$ and $B_F = B'_F/2 = B_g/2 = 0.5$ T ,

we find: $H_F = \dfrac{B_g}{\mu_r \mu_0} \approx 400$ A/m and $H'_F \approx 200$ A/m . We find that $B_g = 1$ T when $I = 16.3$ A.

Inside the ferromagnet $M_F = (\mu_r - 1)H_F \approx 2000 H_F = 8 \cdot 10^5$ A/m . Hence $\sigma_m = \vec{M}_F \cdot \hat{n} = \pm 8 \cdot 10^5$ A/m

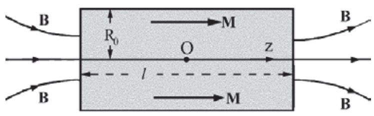
Ampère's equivalence theorem

Currents are sources of B: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$, and so are magnets: $\vec{\nabla}\times\vec{B}=\mu_0\vec{\nabla}\times\vec{M}$

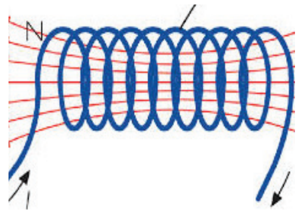
We see that $\vec{\nabla}\times\vec{M}$ acts as a source term for B in the same way as a volume current density. This is the essence of *Ampère's equivalence theorem*: a magnetized body with magnetization density **M**(t,**r**) is equivalent to a current density distribution equal to $\vec{J}_e=\vec{\nabla}\times\vec{M}$
In reality there's also a surface density term $\vec{K}_e=\vec{M}\times\hat{n}$. Examples:

Solenoid ↔ uniformly magnetized bar

(Loop ↔ uniformly magnetized thin disk)

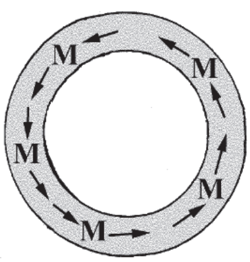


VS

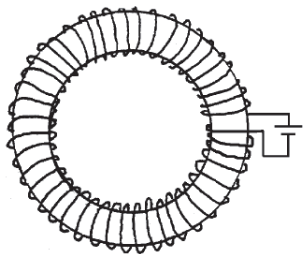


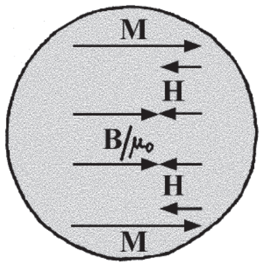
Spherical solenoid ↔ uniformly magnetized sphere

Toroidal solenoid ↔ uniformly magnetized toroid (with/without gap)



VS





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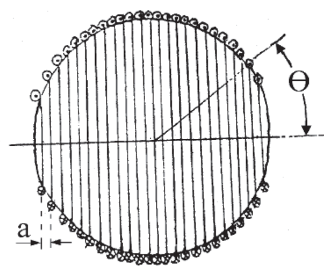


statement of Ampère's theorem:
$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int_{volume} d\tau'\frac{\vec{\nabla}\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}+\frac{\mu_0}{4\pi}\int_{surface} da'\frac{\vec{M}(\vec{r}')\times\hat{n}}{|\vec{r}-\vec{r}'|}$$

Equivalent currents of a magnetized object

Proof of Ampère's equivalence theorem. The vector potential created by a magnetized body is:

$$\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int d\tau'\,\vec{M}(\vec{r}')\times\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}=\frac{\mu_0}{4\pi}\int d\tau'\,\vec{M}(\vec{r}')\times\vec{\nabla}'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)$$

Integrating by parts gives $\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int d\tau'\left[\frac{1}{|\vec{r}-\vec{r}'|}\vec{\nabla}'\times\vec{M}(\vec{r}')-\vec{\nabla}'\times\left(\frac{M(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)\right]$

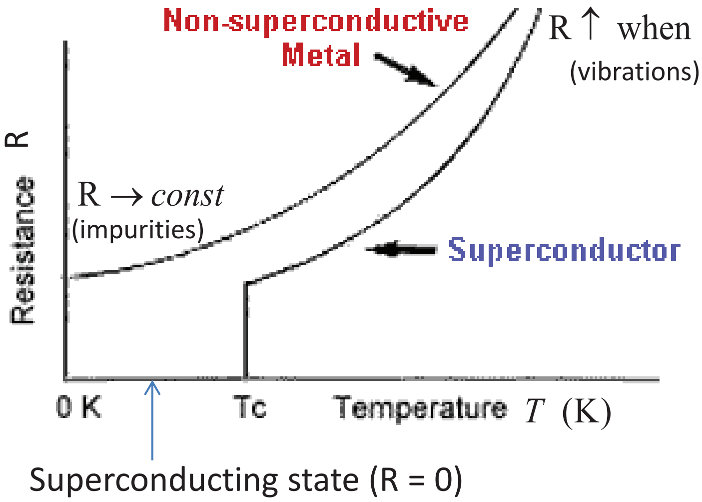
If we define the field $\vec{J}_e=\vec{\nabla}\times\vec{M}$, we see that the 1st term is precisely what we expect for the vector potential of an **(equivalent) volume current density** $\vec{J}_e(\vec{r}')=\vec{\nabla}\times\vec{M}(\vec{r}')$. Moreover, using the vector calculus identity $\int d\tau\,\vec{\nabla}\times\vec{G}=\int da\,\hat{n}\times\vec{G}$ (exercise 0.15(b)), the 2nd term can be rewritten as an integral over the object's surface: $-\frac{\mu_0}{4\pi}\int da'\,\hat{n}\times\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}=\frac{\mu_0}{4\pi}\int da'\,\frac{\vec{M}(\vec{r}')\times\hat{n}}{|\vec{r}-\vec{r}'|}$

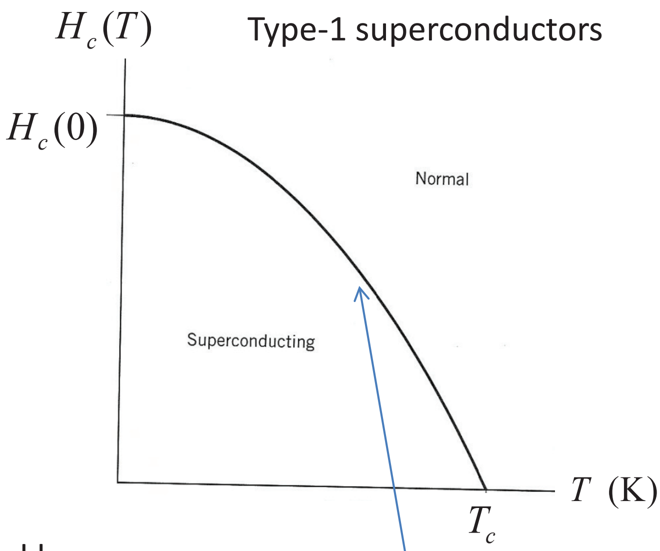
The last expression is formally the vector potential of an **(equivalent) surface current density**, defined as $\vec{K}_e(\vec{r}')=\vec{M}(\vec{r}')\times\hat{n}$

In conclusion we find $\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int d\tau'\frac{\vec{\nabla}\times\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}+\frac{\mu_0}{4\pi}\int da'\frac{\vec{M}(\vec{r}')\times\hat{n}}{|\vec{r}-\vec{r}'|}$

Hence **the vector potential and thus the magnetic B-field generated by a magnetized body of magnetization M(r) are the same as those generated by the set of volume and surface current densities** $\vec{J}_e=\vec{\nabla}\times\vec{M}$ **and** $\vec{K}_e=\vec{M}\times\vec{n}$. These currents are called "EQUIVALENT CURRENTS"

Superconductivity





Superconductivity is characterized by:

1) Zero resistance (R → 0 ⇔ $g \to \infty$)
2) Perfect diamagnetism ($B_{inside} = 0$)

There exists a ***critical temperature*** T_c, above which superconductivity ceases
There exists a ***critical field*** H_c above which superconductivity ceases

It is found experimentally that in type-1 superconductors :

$$H_c(T) \cong H_c(0)\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

Abundance of superconducting elements

30 elements superconduct at ambient pressure, 23 more superconduct at high pressure.

ambient pressure superconductor: T_c(K) / T_c^{max}(K) / P(GPa)

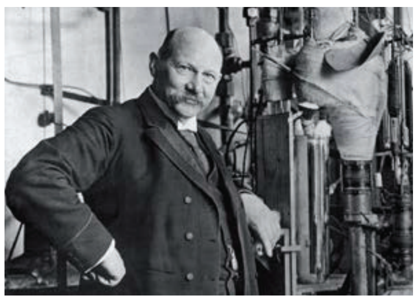
high pressure superconductor: T_c^{max}(K) / P(GPa)

H																	He
Li 0.0004 14 30	Be 0.026											B 11 250	C	N	O 0.6 100	F	Ne
Na	Mg											Al 1.14	Si 8.2 15.2	P 13 30	S 17.3 190	Cl	Ar
K	Ca 25 161	Sc 19.6 106	Ti 0.39 3.35 56.0	V 5.38 16.5 120	Cr	Mn	Fe 2.1 21	Co	Ni	Cu	Zn 0.875	Ga 1.091 7 1.4	Ge 5.35 11.5	As 2.4 32	Se 8 150	Br 1.4 100	Kr
Rb	Sr 7 50	Y 19.5 115	Zr 0.546 11 30	Nb 9.50 9.9 10	Mo 0.92	Tc 7.77	Ru 0.51	Rh .00033	Pd	Ag	Cd 0.56	In 3.404	Sn 3.722 5.3 11.3	Sb 3.9 25	Te 7.5 35	I 1.2 25	Xe
Cs 1.3 12	Ba 5 18	insert La-Lu	Hf 0.12 8.6 62	Ta 4.483 4.5 43	W 0.012	Re 1.4	Os 0.655	Ir 0.14	Pt	Au	Hg-α 4.153	Tl 2.39	Pb 7.193	Bi 8.5 9.1	Po	At	Rn
Fr	Ra	insert Ac-Lr	Rf	Ha													

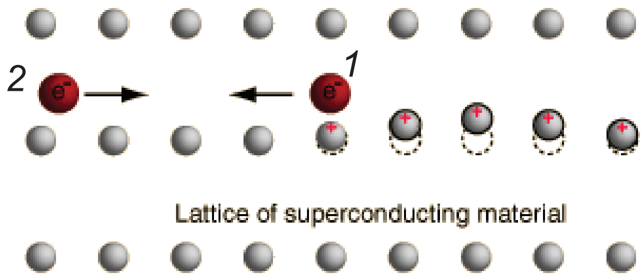
La-fcc 6.00 13 15	Ce 1.7 5	Pr	Nd	Pm	Sm	Eu 2.75 142	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu 12.4 174
Ac	Th 1.368	Pa 1.4	U 0.8(β) 2.4(α) 1.2	Np	Pu	Am 0.79 2.2 6	Cm	Bk	Cf	Es	Fm	Md	No	Lr

High T_c superconductors: discovered in 1986 at IBM

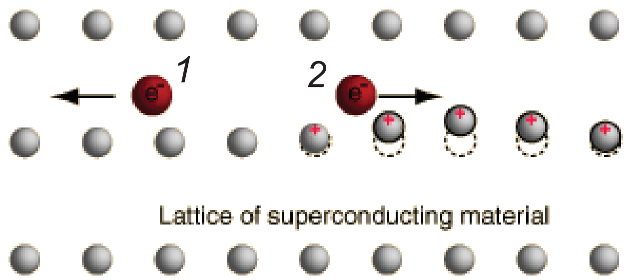
discovery of superconductivity by Heike Kamerlingh Onnes in 1911



Microscopic mechanism of superconductivity



Before:
When electron *1* passes through the lattice, it distorts it (creating a quantized vibration or *phonon*)



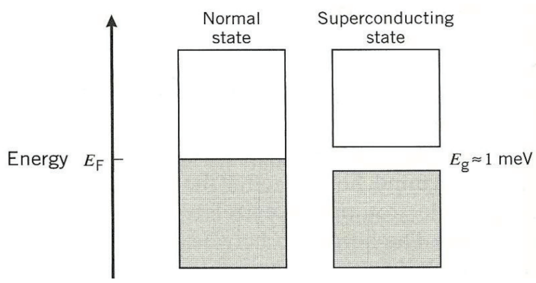
After:
The distortion (*phonon*) creates an excess positive charge that attracts a second electron (*2*) (electron *2* absorbs the phonon excited by *1*)

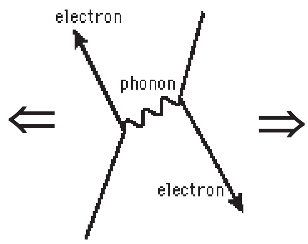
This mechanism results in an effective interaction that brings 2 electrons closer together, forming a so-called ***Cooper pair*** . Assuming that the electrons' speed is given by Fermi' velocity ($v_F = 10^6$ m/s) and that the characteristic time τ of a lattice oscillation is the inverse of the Debye's frequency (see solid state physics course), so that $\tau_D = 10^{-13}$ s , the distance between two electrons in such a pair is approximately: $distance \approx v_F \tau_D \approx 1000 \text{ Å}$

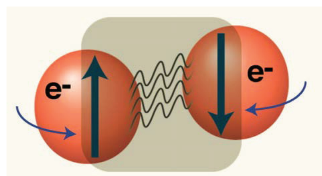
This distance and screening are large enough so as to limit the effect of the Coulomb repulsion

Effects of the effective inter-electron attraction

A metal is characterized by a conduction band that is partially filled up to the so-called "Fermi energy", E_F). When two electrons interact by exchange of a phonon to form a Cooper pair, the energy of the pair is lowered by an amount comparable to the phonon energies, of the order of some meV. This gives rise to a so-called *superconducting gap*, as shown below to the left:







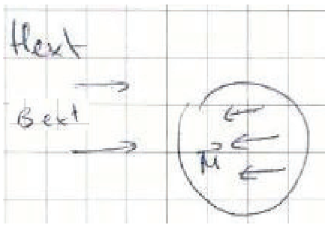
The QM theory of superconductivity is called BCS theory (Bardeen-Cooper-Schrieffer)

Lack of resistance: an electron in a normal metal has energy levels available just above E_F to scatter into; such scattering processes determine a finite resistance. For an electron in a Cooper pair to scatter off a defect, instead, it must break the tie with the other electron, that is, both electrons must gain an energy comparable to half the gap. In other words: *due to the superconducting gap there are no available energy levels for the electrons to scatter into*.
Observation of superconductivity only at low T: at normal temperatures thermal excitations provide enough energy for electrons to jump across the gap
Diamagnetic behavior: the effective coupling is stronger when the two electrons have opposite spins (Cooper pairs are spinless), for the space wavefunction of the pair is then symmetric, which allow them to get closer together. The application of a magnetic field tends to align all electron spins in the same direction thus breaking the Cooper pairs

→ ***superconductivity and magnetism are incompatible with one another***

Superconducting currents

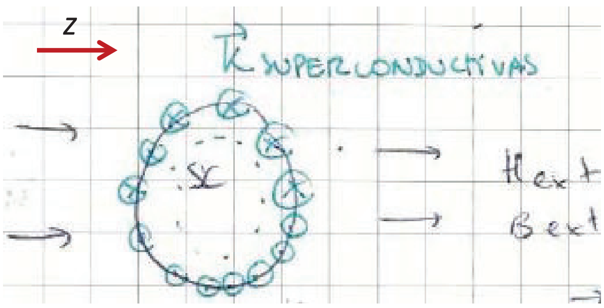
Consider a piece of a type-1 superconductor under an applied external magnetic field. Although a superconductor is nonmagnetic in nature, it responds as a perfectly diamagnetic medium, totally excluding the magnetic field B from its interior. It does so by means of superconducting currents flowing on the superconductor's surface. To calculate the superconducting currents, we use an analogy with equivalent currents. Consider a sphere made of a perfectly diamagnetic material with $\chi_m = -1$, that is, $\mu_r = 0$, so that B = 0 always inside the material.
In an applied field H_{ext}, the sphere will become magnetized so as to screen B from its interior:



$$\vec{M} = \chi_m \vec{H}_{macro} = \chi_m\left(\vec{H}_{ext} + \vec{H}_{resp}\right) = \chi_m\left(\vec{H}_{ext} - \vec{M}/3\right)$$

with $\chi_m = -1$ we get: $\vec{M} - \dfrac{\vec{M}}{3} = -\vec{H}_{ext}$ or: $\vec{M} = -\dfrac{3}{2}\vec{H}_{ext}$

(from this you can verify directly that $\vec{B}_{macro} = \vec{B}_{ext} + \vec{B}_{resp} = 0$)



$\vec{J}_{SC} = 0$ always in a superconductor
(as otherwise $B_{macro} \neq 0$ in the interior)

The response magnetic field is not really produced by a magnetization, since a superconductor is nonmagnetic, but rather by superconducting currents, which we may determine using Ampère's equivalence theorem:

$$\begin{cases} \vec{J}_{SC}\left(=\vec{J}_e\right) = \vec{\nabla}\times\vec{M}_e = 0 \\ \vec{K}_{SC}\left(=\vec{K}_e\right) = \vec{M}_e \times \hat{n} = -\left|\vec{M}_e\right|\hat{z}\times\hat{r} = -\underbrace{\left|\vec{M}_e\right|}_{=\frac{3}{2}H_{ext}}\sin\theta\,\hat{\varphi} \end{cases}$$

Critical current

The existence of a critical field B_c above which superconductivity is destroyed sets an upper limit to the current that a type-1 superconducting wire can carry. By Ampère's law, a current I in a long wire of radius R_0 generates a magnetic field of magnitude $B = \dfrac{\mu_0 I}{2\pi s}$, for $s \geq R_0$, where s is the radial distance from the wire axis. This result does not depend on how the current is distributed as long as it is symmetrical. In type-1 superconductors there can be no field inside the material; hence the current must be distributed on the outer surface of the wire. The critical current is to the value of I at which the field at the wire's surface is equal to the critical value:

$$I_c = \frac{2\pi R_0}{\mu_0} B_c$$

Despite being confined to the superconductor's outer surface, critical currents can be surprisingly large. For example, consider a wire made of tin with a radius about the size of a hair (40 μm). How much current can the wire carry at zero kelvin and remain superconducting?

Answer: critical magnetic field of tin at 0 K: 0.03 T . The maximum superconducting current is:

$$I_c = \frac{2\pi R_0 B_c}{\mu_0} = \frac{2\pi(4\cdot 10^{-5})(0.03)}{4\pi\cdot 10^{-7}} = 6\ \text{A}$$

Examples of superconductors with very high critical fields:

superconductor	$Nb_{0.75}Zr_{0.25}$	Nb_3Sn	$V_{2.95}Ga$
B_c (T)	11	20	35

Applications of superconductors

- Superconducting magnetic levitation train SCMaglev (Japan, based on the Meissner effect)
http://en.wikipedia.org/wiki/SCMaglev

- powerful electromagnets (with NbTi or Nb_3Sn wires) → B up to 20 T
they are for example used in magnetic resonance imagning (MRI)

- storage of energy into superconducting currents, energy transmission at a distance with superconducting cables (no losses due to Joule heating)

- high-precision B-field measurements (SQUID devices)

- further miniaturization of electronic chips (size is now limited due to heating by Joule effect)

Multipole expansion for **A**

The vector potential for a current distribution is $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$

We carry out the same analysis that we employed for the electrostatic potential. Since:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = [(\vec{r}-\vec{r}\,')\cdot(\vec{r}-\vec{r}\,')]^{-\frac{1}{2}} = \left[r^2 - 2\vec{r}\cdot\vec{r}\,' + r'^2\right]^{-\frac{1}{2}} = \frac{1}{r}\left(1 - 2\frac{\hat{r}\cdot\vec{r}\,'}{r} + \frac{r'^2}{r^2}\right)^{-1/2}$$

Using the Taylor expansion $(1+\varepsilon)^{-\frac{1}{2}} \cong 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 + \ldots$, we can calculate the vector potential produced by a general distribution of currents at large distance *r* as:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d\tau'\,\vec{J}(\vec{r}\,')\Big(1 - \underbrace{\frac{2\hat{r}\cdot\vec{r}\,'}{r} + \frac{r'^2}{r^2}}_{\varepsilon}\Big)^{-\frac{1}{2}} = \frac{\mu_0}{4\pi\, r}\int d\tau'\,\vec{J}(\vec{r}\,')\left[1 + \frac{\hat{r}\cdot\vec{r}\,'}{r} - \frac{1}{2}\frac{r'^2}{r^2} + \frac{3}{8}\left(\frac{\cdots}{r^2} + \frac{\cdot\cdot}{r^3} + \frac{\cdots}{r^4}\right) + ..\right]$$

The first term in this multipole expansion, proportional to 1/r , is zero because the total current of a set of moving charges confined to a finite region of space must be zero: $\int d\tau'\,\vec{J}(\vec{r}\,') = 0$ The first non-zero term, that scales as $1/r^2$, is the so-called "magnetic dipole" term, given by:

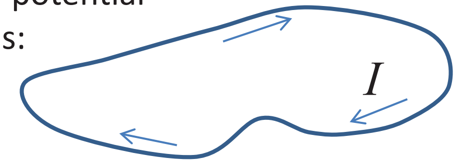
$$\vec{A}(\vec{r}) \approx \frac{\mu_0}{4\pi\, r^2}\int d\tau'\,\vec{J}(\vec{r}\,')\hat{r}\cdot\vec{r}\,' \propto \frac{1}{r^2}$$ **(magnetic) dipole term**

We see that the leading term in A is $1/r^2$ (just as for a spin magnetic moment). Therefore, the leading order in B must be proportional to $1/r^3$: $\vec{A}(\vec{r}) \sim \frac{1}{r^2} + \ldots \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \vec{B}(\vec{r}) \sim \frac{1}{r^3} + \ldots$

Hence **the B-field of a delimited current distribution goes at the most as $1/r^3$** (this is not true for an infinite straight wire or an infinite current plane, which are not delimited currents)

the backside of Ampère's equivalence theorem: equivalent magnetic dipole of a current loop

For a closed loop run by a current I, the leading term of the vector potential at large distances can be written, using the equality $\vec{J}d\tau = Id\vec{\ell}$, as:



$$\vec{A}(\vec{r}) \approx \frac{\mu_0}{4\pi\, r^2}\int d\tau'\, \vec{J}(\vec{r}')\hat{r}\cdot\vec{r}' = \frac{\mu_0 I}{4\pi r^2}\oint d\vec{\ell}'(\hat{r}\cdot\vec{r}')$$

By making use of the vector calculus theorem (Pr. 0-15(a)): $\oint_{loop} T d\vec{\ell} = \int_{surface} d\vec{a}\times(\vec{\nabla}T)$, we get:

$$\vec{A}(\vec{r}) \approx \frac{\mu_0 I}{4\pi r^2}\oint d\vec{\ell}'(\hat{r}\cdot\vec{r}') = \frac{\mu_0 I}{4\pi r^2}\int d\vec{a}'\times\left[\vec{\nabla}'(\hat{r}\cdot\vec{r}')\right]$$

If $\vec{C}$ is independent of $\vec{r}'$, it is $\vec{\nabla}'(\vec{C}\cdot\vec{r}') = (C_x, C_y, C_z) = \vec{C}$. Hence we get $\vec{\nabla}'(\hat{r}\cdot\vec{r}') = \hat{r}$

Thus: $\vec{A}(\vec{r}) \approx \frac{\mu_0 I}{4\pi r^2}\oint d\vec{a}'\times\hat{r} = \frac{\mu_0}{4\pi r^3}\left(I\oint d\vec{a}'\right)\times\vec{r} = \frac{\mu_0}{4\pi r^3}\vec{m}_{eq}\times\vec{r}$

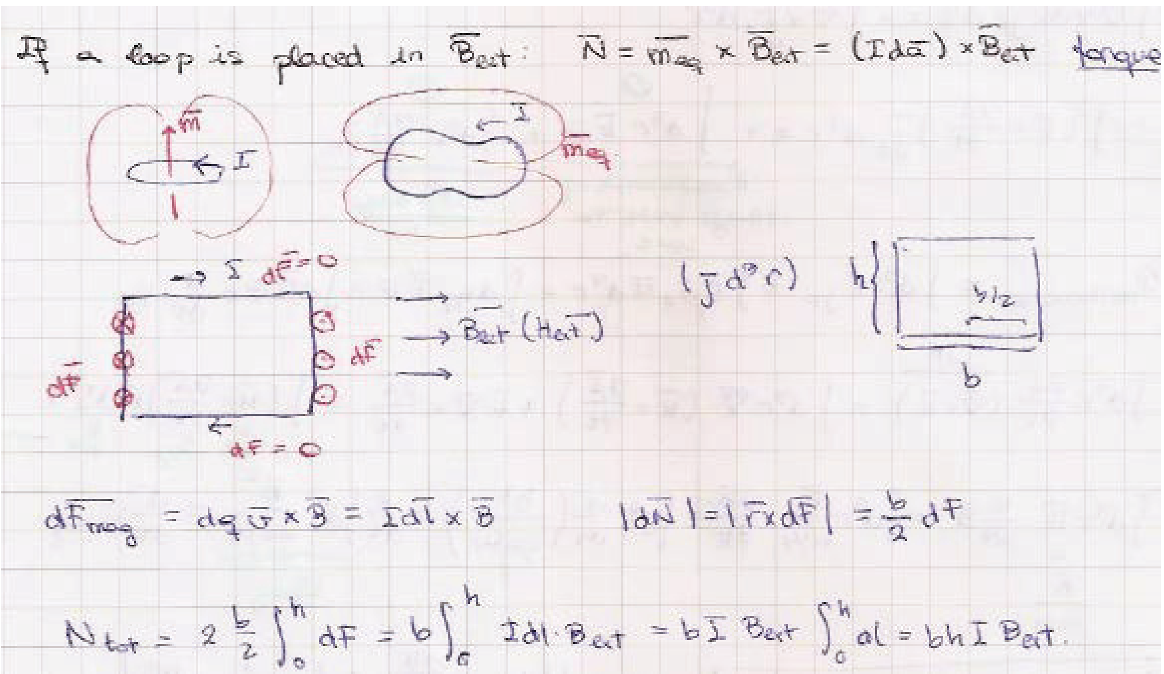
We see here that the vector potential of a loop is, in this approximation, the same as that of an equivalent point magnetic dipole: $\vec{m}_{eq} = I\oint d\vec{a}'(= I\vec{a})$ ***equivalent magnetic dipole of a current loop***

The quantity $\vec{a} = \oint d\vec{a}'$ is called the "vector area". For a flat loop, it is simply the area enclosed by the loop times the normal direction: $\vec{m}_{eq} = Ia\hat{n}$ ***equivalent magnetic dipole of a flat loop***

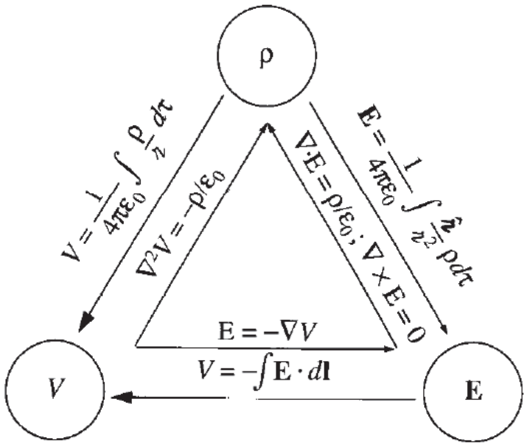
Magnetic dipole of a planar current loop

The vector potential and thus the B-field created by a current loop is (in 1st order approximation) the same as that of a spin: $\vec{A}(\vec{r}) \approx \frac{\mu_0}{4\pi r^3}\vec{m}_{eq}\times\vec{r}$ with $\vec{m}_{eq} = I\oint d\vec{a}' \quad (= I\vec{a})$

This analogy between current loops and spins is much deeper. For example, we can calculate the torque on a loop under an applied field, or the energy of (and force on) a loop in an external (non-uniform) field in the same way as we do for a magnetic dipole. Here is an example:



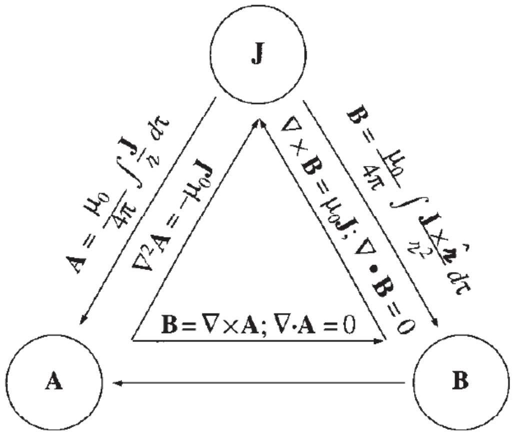
summary of Electrostatics & Magnetostatics



Lorentz force:

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\mathcal{E}_{\substack{closed\\loop}} = -\frac{d\Phi}{dt}$$



$\nabla \times \mathbf{E} = 0$	*Maxwell's equations:*	$\nabla \times \mathbf{H} = \mathbf{J}$
$\nabla \cdot \mathbf{D} = \rho_f$	← *Electrostatics* *Magnetostatics* →	$\nabla \cdot \mathbf{B} = 0$

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

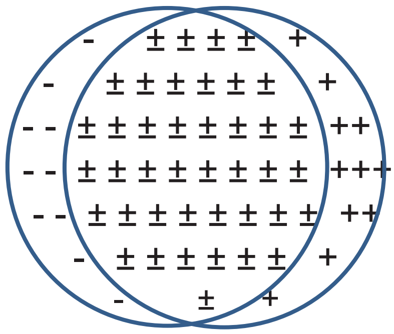
$$\rho_b = -\vec{\nabla}\cdot\vec{P}\ ;\ \sigma_b = \vec{P}\cdot\hat{n} \qquad \vec{\nabla}\cdot\vec{J}_f = 0 \qquad \rho_m = -\vec{\nabla}\cdot\vec{M}\ ;\ \sigma_m = \vec{M}\cdot\hat{n}$$

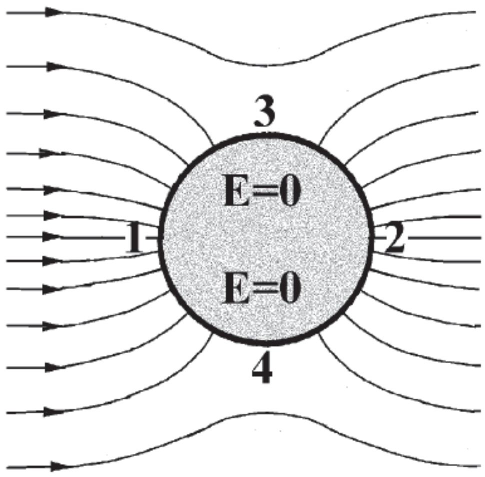
Linear media: $\vec{P} = \chi_{el}\varepsilon_0\vec{E}$ $\quad$ $\vec{J}_f = g\vec{E}$ $\quad$ $\vec{M} = \chi_m\vec{H}$

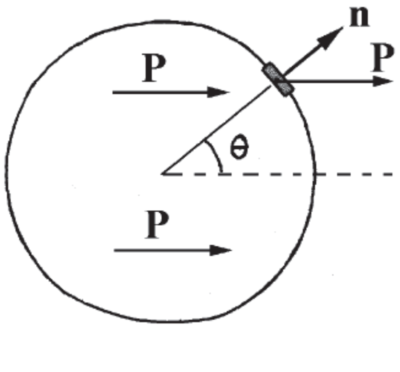
$\vec{D} = \varepsilon_r\varepsilon_0\vec{E}$ $\quad$ $\vec{B} = \mu_r\mu_0\vec{H}$

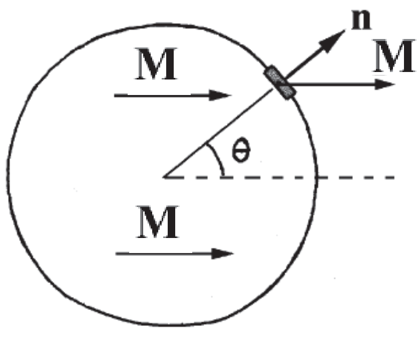
Linear homogeneous medium: $\rho_f = \varepsilon_r\rho_{tot}$ $\quad$ $RC = \varepsilon_r\varepsilon_0/g$

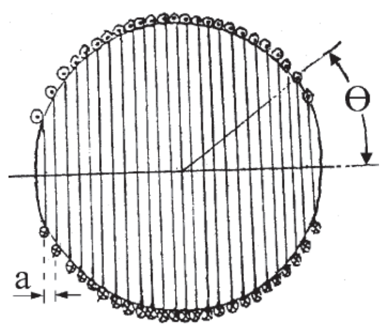
Analogies between plasmas, metals, dielectrics, magnetic media, & current distributions











https://www.youtube.com/watch?v=sauxaWJynHo