Time-varying fields & Maxwell's equations (TOPIC 5)

beyond the steady state

A time varying magnetic field generates an electric field ("induction"), and a time-varying electric field generates a magnetic field ("displacement current")

The first concept was demonstrated by one of the greatest experimenters of all time, the second one was first introduced by one of the greatest mathematical physicists of history

Why do we study in detail time-dependent fields?

1) Only with time-varying fields it is possible to derive the formula of the magnetic field energy
2) Only with time-varying fields is the connection between electricity and magnetism evident
3) Electric machines (generators, motors and transformers) work with time-varying fields
4) Time-varying charges and currents generate electromagnetic radiation:
   light is a time-varying electric & magnetic field (electromagnetic wave)

Contents of TOPIC 5:

- Electromotive force with static & time-varying magnetic fields: Lorentz-force; Faraday's law in integral & differential form; relation Lorentz force & Faraday's law
- Magnetic energy and magnetic forces, applications: generator, motor, transformer
- Time varying electric fields: Ampère-Maxwell's law and displacement currents
- Maxwell's equations, Poynting vector and Poynting's theorem, irradiance
- Electromagnetic waves, refractive index for perfect insulators, Snell's law

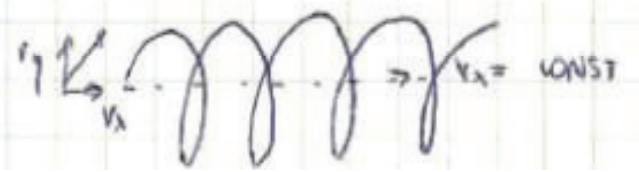
Lorentz force law & Electromotive forces

Lorentz force on moving charges $\vec{F}_{Lorentz} = q\vec{E} + q\vec{v}\times\vec{B}$

Lorentz force "density" : (force per unit volume)

$$\vec{f}(t,\vec{r}) = \rho(t,\vec{r})\left[\vec{E}(t,\vec{r}) + \vec{v}_{drift}(t,\vec{r})\times\vec{B}(t,\vec{r})\right] = \rho(t,\vec{r})\vec{E}(t,\vec{r}) + \vec{J}(t,\vec{r})\times\vec{B}(t,\vec{r})$$

Example: motion in (quasi) uniform B (and E) fields

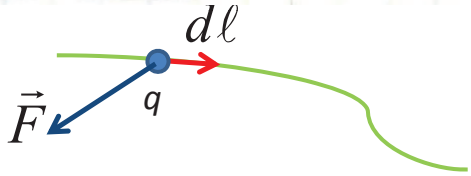


Polar aurora:

DEF: electromotive force = (electromotance, *emf* , = work per unit charge)

$$\mathscr{E} = \int \vec{f}\cdot d\vec{\ell} = \frac{1}{q}\int \vec{F}\cdot d\vec{\ell}$$

force per unit charge



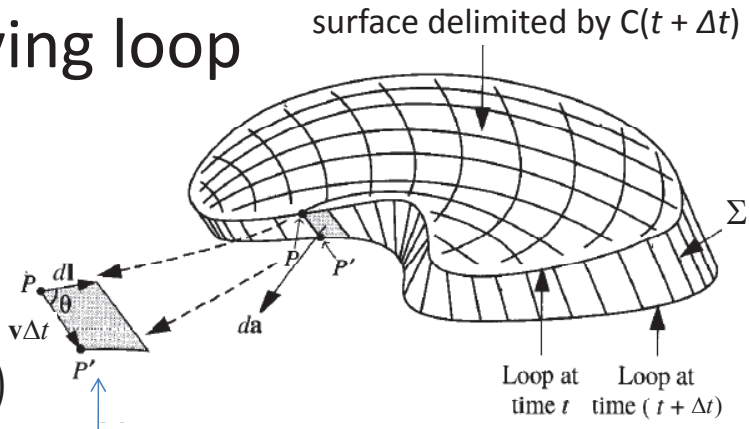
If the applied force is the Lorentz force:

$$\mathscr{E} = \int \vec{f}\cdot d\vec{\ell} = \int \frac{\vec{F}_{Lorentz}}{q}\cdot d\vec{\ell} = \int \vec{E}\cdot d\vec{\ell} + \int \left(\vec{v}\times\vec{B}\right)\cdot d\vec{\ell} = \underbrace{\mathscr{E}_{elec}}_{=-\Delta V} + \mathscr{E}_{mag}$$

$\Rightarrow$ there are 2 ways to generate a current:
- **electric electromotance: batteries & fuel cells**
- **magnetic electromotance: generators**

IMPORTANT : the B-field does no work on moving charges, but it does work on magnetic dipoles (and thus on magnetic materials). Moreover, energy must be spent to create magnetic fields.

Magnetic *emf* in a moving loop



$$\vec{F}_{mag} = q\vec{v}_{\text{charge}} \times \vec{B} = q\left(\vec{v}_{\text{drift}} + \vec{v}_{\text{wire}}\right) \times \vec{B}$$

$\boldsymbol{v}_{\text{drift}}$ is the speed of the charge with respect to the wire: it is parallel to $\boldsymbol{j}$ and thus to $d\boldsymbol{\ell}$

$$\mathscr{E} = \mathscr{E}_{mag} = \oint_{loop} d\vec{\ell} \cdot \left[\left(\vec{v}_{\text{drift}} + \vec{v}_{\text{wire}}\right) \times \vec{B}\right] =$$

$$= \oint_{loop} \vec{B} \cdot \left[d\vec{\ell} \times \left(\vec{v}_{\text{drift}} + \vec{v}_{\text{wire}}\right)\right] = \oint_{loop} \vec{B} \cdot \left(d\vec{\ell} \times \vec{v}_{\text{wire}}\right)$$

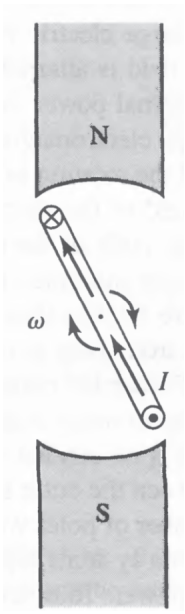
$d\vec{\ell} \times \vec{v}_{wire} = d\vec{\ell} \times \Delta\vec{r}/\Delta t$, where $\Delta\vec{r}$ is the displacement of wire segment $d\vec{\ell}$ during Δt

But $d\vec{\ell} \times \Delta\vec{r} = \hat{n}dA$ is the outward directed area element of a section of the ribbon. Hence: $\mathscr{E} = \Phi(\Sigma)/\Delta t$, where $\Phi(\Sigma)$ is the flux of B through the lateral surface Σ (the integration is only performed on $d\boldsymbol{\ell}$ since in the other direction the thickness is infinitesimal). Now, the flux of B through a surface enclosing a volume must be zero; hence $\Phi(\Sigma) + \Phi(C(t + \Delta t)) + \Phi(C(t)) = 0$, where the 2nd and 3rd terms are the flux through the upper and lower surfaces in the figure. In the definition of the flux, the direction of $\hat{n}$ is outwards. But here it is more convenient to define it always in the same direction with respect to the loop. Let's say it is outwards (parallel to $\hat{n}$) for the surface S(t + Δt); then it is antiparallel to $\hat{n}$ for S(t); with this definition the sign of $\Phi(C(t))$ must be changed, to give: $\mathscr{E} = \Phi(d\Sigma)/dt = -\dfrac{\Phi(C(t+dt)) - \Phi(C(t))}{dt} = -\dfrac{d\Phi(C(t))}{dt}$

This magnetic electromotive force (which does not stem from any battery/voltage difference) is non-conservative (the line integral of $\mathbf{F}_{mag}$ around the closed loop is different from zero).

Applications of Lorentz's law and magnetic *emf*

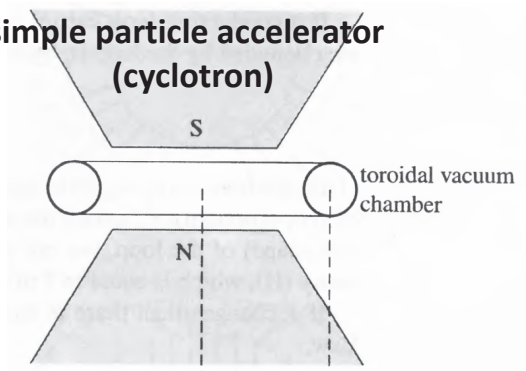
How can one tell which way a magnetic emf goes? Look at the direction of the magnetic force, or remember Lenz's law: **the current generated by a magnetic *emf* always flows so as to oppose the change in external flux**.



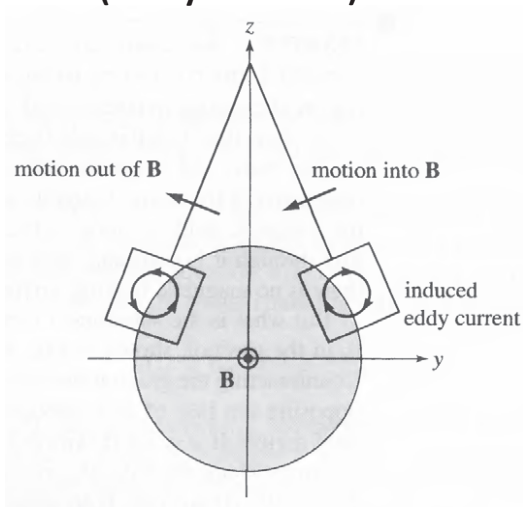
AC generator & electric motor

$$\Phi = \int B \cdot dA = BA\cos\omega t$$
$$\Rightarrow \mathscr{E} = BA\omega \sin \omega t$$

simple particle accelerator (cyclotron)



Magnetic brake ("Eddy currents")

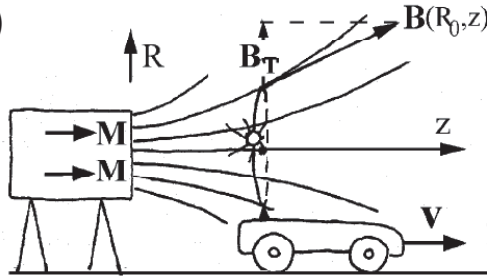


Other applications of the Lorentz force:

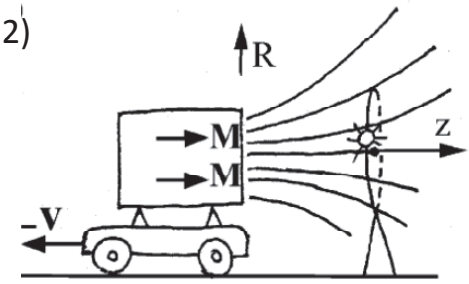
- DC electric motor
- mass spectrometer
- Hall effect (metal/semiconductor) & polarization of dielectric moving in B-field

Faraday's law: induction

(1)



(2)



The two situations (1) & (2) are the same, only described by observers moving with relative speed **v** :

-In (1), the charges in the loop are moving in a static B-field and are subject to the Lorentz force, which acts as electromotive force (topic 4): $\boxed{\mathscr{E} = -\dfrac{d\Phi_B}{dt}}$

-In (2) the same net electromotive force must be present → the very existence of the Lorentz force by itself implies a modification of the field equations for **E**: the line integral of E cannot be zero in (2), it must be equal to: $\boxed{\oint \vec{E}\cdot d\vec{\ell} = -\dfrac{d\Phi_B}{dt}}$

In order for the equation of the e.m.f to holds for case (2) no matter what the shape of the loop, it must be:

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$

Faraday's law: <u>a changing magnetic field produces an electric field</u> (& thus an induced e.m.f)

In the most general case, the total e.m.f. in a moving loop which is not connected to a power supply is the sum of two terms, one due to the Lorentz force, and the other due to induction:

$\mathscr{E} = -\int_S \dfrac{\partial\vec{B}}{\partial t}\cdot d\vec{a} + \oint_{loop}\left(\vec{v}\times\vec{B}\right)\cdot d\vec{\ell}$. The total e.m.f. is always equal to: $\mathscr{E} = -\dfrac{d\Phi}{dt} = -\dfrac{d}{dt}\left(\int_S \vec{B}\cdot d\vec{a}\right)$

Implications of Faraday's law

In the most general case, the total e.m.f. in a moving loop which is not connected to a power supply is the sum of two terms, one due to the Lorentz force, and the other due to induction:

$\mathscr{E} = -\int_S \dfrac{\partial\vec{B}}{\partial t}\cdot d\vec{a} + \oint_{loop}\left(\vec{v}\times\vec{B}\right)\cdot d\vec{\ell}$. The total e.m.f. is always equal to: $\mathscr{E} = -\dfrac{d\Phi}{dt} = -\dfrac{d}{dt}\left(\int_S \vec{B}\cdot d\vec{a}\right)$

electromagnetic induction = generation of an E-field by a B-field that varies with time

Faraday's law $\quad \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$

(actually discovered independently by Faraday and Henry in 1830-1831)

Simple application: Find the time-varying **B** field that induces the electric field:

$\vec{E} = (ax^2y^3\hat{x} + \beta z^4\hat{y})\cos\omega t$ $\quad$ Answer: $\vec{B} = (4\beta z^3\hat{x} + 3ax^2y^2\hat{z})\dfrac{\sin\omega t}{\omega}$

Since $\vec{B} = \vec{\nabla}\times\vec{A}$, we can write Faraday's law as: $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\left(\vec{\nabla}\times\vec{A}\right)}{\partial t}$ or: $\vec{\nabla}\times\left(\vec{E} + \dfrac{\partial\vec{A}}{\partial t}\right) = 0$

The last equality implies that the field in brackets is a conservative field, hence one can write:

$$\boxed{\vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\mathrm{V}}$$

This equation has two extremely important consequences:

1) E is no longer conservative: $\vec{E} = -\vec{\nabla}\mathrm{V} - \dfrac{\partial\vec{A}}{\partial t}$

2) In the presence of time-variations, the energy U = qV associated with electrical interactions e.g. inside a battery, goes not only in the creation of **electrical** fields, but also of **magnetic** fields

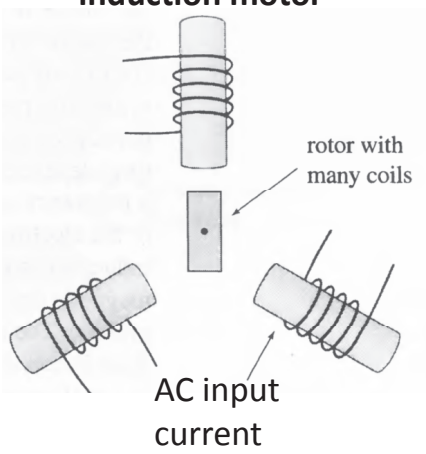
Technological applications of Faraday's law

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

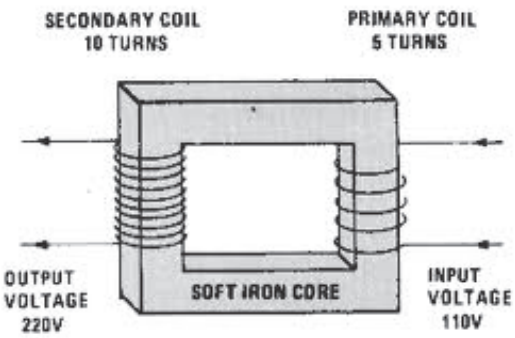
Applications of Faraday's discoveries:

1) induced emf
2) Magnetic energy and self-inductance, inductors
3) Foucault (or Eddy) currents: induction stove (heating due to Joule effect and hysteresis loss of ferromagnetic iron pan which also amplifies the magnetic field – you need a special pan for an induction kitchen!)
4) induction motor, transformer, …

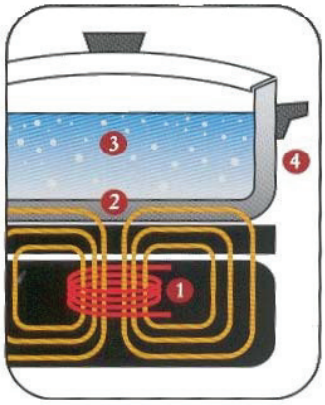
induction motor



transformer



induction cooker



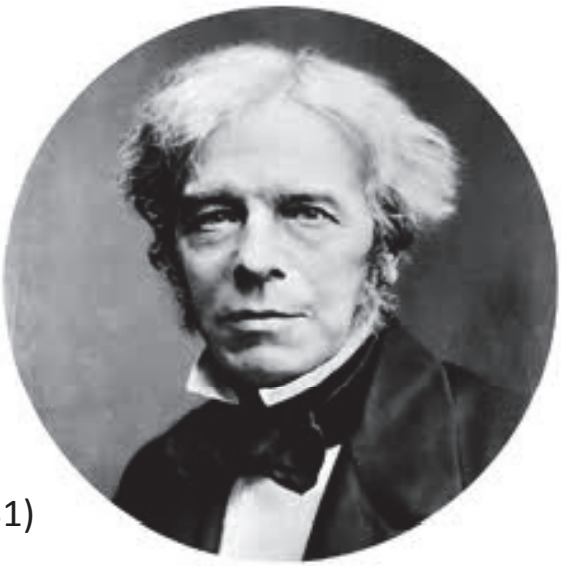
Michael Faraday's curriculum vitae

Scientific terms and concepts introduced by M.F. :

Force field, **force lines** (electric and magnetic lines), **ion** (anion, cation), **voltmeter**, **electrode** (anode/cathode), electrolyte, **electrolysis**, **dielectric**, **dia- & paramagnetic**

Most important scientific discoveries

1) Faraday's **induction** law
2) Linear dielectric and magnetic response:
   - **diamagnetism**
   - **dielectric constant**
3) Faraday's laws of **electrochemistry**
4) Faraday rotation (**magneto-optic** effect)



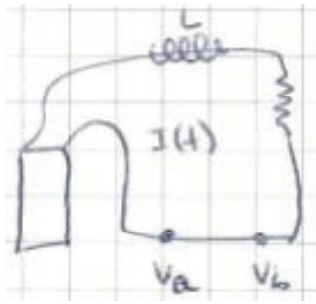
Inventions

- Electric motor (1821)
- Electric generator and precursor of dynamo (1831)
- precursor of transformer (1831)

Units named after M.F.:

farad (capacitance) ; **faraday** (electrochemistry: charge of N_A electrons)

Energy balance in a circuit



Consider a circuit in which the current density is increased from zero to some value $\mathbf{J}_f$. The power source (*e.g.* battery) needs to supply energy to accelerate the electrons; such energy is not only to overcome friction, but also the effect of the opposing Faraday field generated by the increment of **J**, that is, the corresponding variation of Φ.

Take an element dℓ of the circuit, of cross section da . Let's calculate the work done by the source (battery) to increase the current across dℓ from zero to a finite value I_f . The source maintains a fixed voltage drop across any two points of the circuit. If at a given instant the current through the circuit is I(t), the instantaneous power supplied by the source is equal to :

$$\wp_{battery} = -(\mathrm{V}_b - \mathrm{V}_a)I(t) = -\Delta \mathrm{V} I$$

If the two points are an infinitesimal distance dℓ away, the voltage drop between them is dV, which is the variation of the scalar potential field across dℓ: $d\mathrm{V} = \nabla \mathrm{V} \cdot d\ell$

Hence, using $I d\vec{\ell} = \vec{J}_f d\tau$, we get: $d\wp = -(d\mathrm{V})I = -\vec{\nabla}\mathrm{V} \cdot d\vec{\ell}\, I = -\vec{\nabla}\mathrm{V} \cdot \vec{J}_f\, d\tau$

Now, we saw that Faraday's law implies $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\mathrm{V}$. We thus get:

$$d\wp = \vec{E} \cdot \vec{J}_f\, d\tau + \frac{\partial \vec{A}}{\partial t} \cdot \vec{J}_f\, d\tau \Rightarrow \qquad \wp = \underbrace{\int d\tau\, \vec{E} \cdot \vec{J}_f}_{\text{(1) mechanical power } \wp_{\text{charges}}} + \underbrace{\int d\tau\, \frac{\partial \vec{A}}{\partial t} \cdot \vec{J}_f}_{\text{(2) Variation of magnetic energy}}$$

The first term is the mechanical power absorbed by the charges, which is finally dissipated as Joule heat. The second one is instead stored as **magnetic energy** as we show in the next slide →

Magnetic energy

$$\wp = \underbrace{\int d\tau\, \vec{E} \cdot \vec{J}_f}_{\text{(1) } \wp_{\text{charges}}} + \underbrace{\int d\tau\, \frac{\partial \vec{A}}{\partial t} \cdot \vec{J}_f}_{\text{(2) } dU_{\text{mag}}/dt}$$

The first term is the mechanical power required to accelerate the charges and keep them moving against the viscous friction drag responsible for the resistance of the wire:

$$\text{(1)} \int d\tau\, \vec{E} \cdot \vec{J}_f = \int d\tau\, \vec{E} \cdot \rho_f \vec{v}_{drift} \underset{\rho_f d\tau = dq_f}{=} \left(\int dq_f \vec{E}\right) \cdot \vec{v}_{drift} = \left(\int d\vec{F}\right) \cdot \vec{v}_{drift} = \vec{F} \cdot \vec{v}_{drift} = \wp_{\text{charges}}$$

Notice that we can extend the integrals to all space, since $\vec{J}_f$ and $\vec{v}_{drift}$ are zero outside the wire

The 2nd term is the power used up against the induced e.m.f. created by the time-variation of the magnetic field. It is the time-variation of a **magnetic energy** stored in the magnetic field:

$$\text{(2)} \int d\tau \frac{\partial \vec{A}}{\partial t} \cdot \vec{J}_f = \int d\tau \frac{\partial \vec{A}}{\partial t} \cdot \left(\vec{\nabla} \times \vec{H}\right) = \underbrace{\int d\tau\, \vec{\nabla} \cdot \left(\vec{H} \times \frac{\partial \vec{A}}{\partial t}\right)}_{= \int \left(\vec{H} \times \frac{\partial \vec{A}}{\partial t}\right) \cdot d\vec{a} = 0 \text{ at } \infty} + \int d\tau\, \vec{H} \cdot \left(\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}\right) = \int d\tau\, \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

In linear media $\vec{B} = \mu_r \mu_0 \vec{H}$. Hence:

$$(2) = \int d\tau \frac{\vec{B}}{\mu_r \mu_0} \cdot \frac{\partial \vec{B}}{\partial t} = \int d\tau \frac{d}{dt}\left(\frac{B^2}{2\mu_r \mu_0}\right) = \frac{d}{dt}\int d\tau \frac{B^2}{2\mu_r \mu_0} = \frac{d}{dt}\int d\tau\, u_{mag} = \frac{dU_{mag}}{dt}$$. Therefore:

$$\wp_{source} = \wp_{\text{charges}} + \frac{dU_{\text{mag}}}{dt} \quad \text{with} \quad U_{mag} = \int d\tau\, u_{mag} \quad \text{and} \quad u_{mag} = \frac{B^2}{2\mu_r \mu_0} = \frac{1}{2}\vec{H} \cdot \vec{B}$$

U_{mag} is the energy stored in the magnetic field (= used to build up the field against induction)

Self-inductance and magnetic energy

The magnetic energy in linear media can also be written as $U_{mag}=\frac{1}{2}\int d\tau\, \vec{J}_f\cdot\vec{A}$. In fact it is:

$$U_{mag}=\frac{1}{2}\int d\tau\, \vec{H}\cdot\vec{B}=\frac{1}{2}\int d\tau\, \vec{H}\cdot\left(\vec{\nabla}\times\vec{A}\right)=\frac{1}{2}\int \underbrace{d\tau\, \vec{\nabla}\cdot\left(\vec{A}\times\vec{H}\right)}_{=\frac{1}{2}\int\left(\vec{A}\times\vec{H}\right)\cdot d\vec{a}=0 \text{ at } \infty}+\frac{1}{2}\int d\tau\, \vec{A}\cdot\left(\vec{\nabla}\times\vec{H}\right)=\frac{1}{2}\int d\tau\, \vec{A}\cdot\vec{J}_f$$

According to Biot-Savart's law, the B-field of a loop carrying a uniform current I is:

$$\vec{B}_{\text{loop}}=\frac{\mu_0}{4\pi}\oint\frac{d\tau'\,\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}=\frac{\mu_0}{4\pi}\oint\frac{I d\vec{\ell}\,'\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

We see that the field is linearly proportional to I. In the same way, the flux of B through the surface delimited by the loop is also proportional to the current:

$$\Phi_B=\int\vec{B}\cdot d\vec{a}=\frac{\mu_0}{4\pi}I\int\left(\oint\frac{d\vec{\ell}\,'\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right)\cdot d\vec{a}=L\,I$$

$$\boxed{\Phi_B=L\,I}$$

The proportionality coefficient L between the flux of B and I is the (*self-*)*inductance* of the closed loop. In terms of *L*, Faraday's law becomes:

$$\mathscr{E}=-\frac{d\Phi_B}{dt}=-L\frac{dI}{dt}$$

(this is the formula you use in circuit theory)

Similarly, we find for the magnetic energy of a loop (using $I d\vec{\ell}=\vec{J}_f d\tau$):

$$U_{mag}=\frac{1}{2}\int d\tau\,\vec{J}_f\cdot\vec{A}=\frac{1}{2}I\oint d\vec{\ell}\cdot\vec{A}=\frac{1}{2}I\int\left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{a}=\frac{1}{2}I\int\vec{B}\cdot d\vec{a}=\frac{1}{2}I\,\Phi_B$$

Stokes' theorem

$\Rightarrow$ **Magnetic energy of a loop:** $U_{mag}=\frac{1}{2}I\,\Phi_B=\frac{1}{2}LI^2=\frac{\Phi_B{}^2}{2L}$

Magnetic vs electric energy formulas

It is interesting to compare the energy formulas we just found with those for electrostatics:

$$u_{mag}=\frac{1}{2}\vec{B}\cdot\vec{H}=\frac{1}{2}\vec{J}_f\cdot\vec{A}$$

$$u_{el}=\frac{1}{2}\vec{D}\cdot\vec{E}=\frac{1}{2}\rho_f V$$

$$L=\frac{\Phi}{I_f}\qquad U_{mag}=\frac{1}{2}\Phi I=\frac{1}{2}LI^2$$

$$C=\frac{Q_f}{V}\qquad U_{el}=\frac{1}{2}QV=\frac{1}{2}CV^2$$

$$U=-\vec{m}\cdot\vec{B}_{ext}$$

$$U=-\vec{p}\cdot\vec{E}_{ext}$$

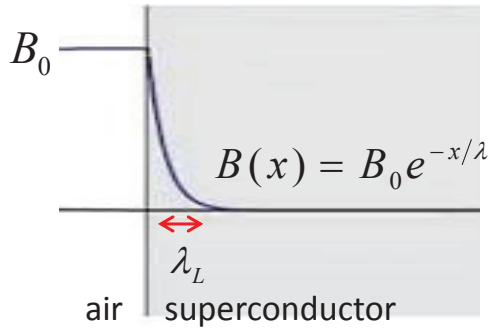
*Superconductors: London eq., Meissner effect

Superconducting currents are not really two-dimensional: they are not strictly confined to the surface of the medium but actually penetrate inside it a short distance, called *London penetration depth*. In a normal metal, Ohm's law $\vec{J}_f = g\vec{E}$ holds. The fact that **J** (a constant times $\mathbf{v}_{\text{drift}}$) is proportional to **E** is a consequence of electron collisions. But in a superconductor such collisions do not occur; rather, electrons have an accelerated motion: $a_d = \frac{dv_d}{dt} = \frac{eE}{m}$

Since $\frac{\partial J}{dt} = ne\frac{dv_d}{dt}$ (where *n* = density of superconducting electrons) we get, instead of Ohm's law, the *London equation*: $\frac{\partial J}{\partial t} = \frac{ne^2 E}{m} = \kappa E$ (with $\kappa = \frac{ne^2}{m}$). Taking the curl of both sides and using Faraday's law we obtain: $\frac{\partial(\nabla \times J)}{\partial t} = \kappa \nabla \times E = -\kappa \frac{\partial B}{\partial t}$. Hence apart from a constant that we can discard, we have $\vec{\nabla} \times \vec{J} = -\kappa \vec{B}$. Combining this with Ampère's law for B, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ (valid since a superconductor is nonmagnetic), we get:

$\vec{\nabla} \times \vec{\nabla} \times \vec{B} = -\mu_0 \kappa \vec{B}$, which yields, using $\vec{\nabla} \cdot \vec{B} = 0$:

$$\nabla^2 \vec{B} = \mu_0 \kappa \vec{B} \equiv \frac{\vec{B}}{\lambda_L^2} \qquad \text{with} \quad \lambda_L = \sqrt{\frac{m}{\mu_0 n e^2}} = \text{skin depth}$$



The solution to this equation is (in 1D) : $B(x) = B_0 e^{-x/\lambda}$
This shows that B decays inside a superconductor in a short distance λ_L . Hence B = 0 inside : this is the Meissner effect !

*Condensation energy of a superconductor

We saw in topic 4 that a magnetic field destroys the superconducting state. We can use this to calculate the energy density of the superconducting state at a given temperature *T* . Take a long cylinder made of a superconducting material, initially in the absence of a magnetic field. As we turn on a weak magnetic field, it fills all space except for the region occupied by the cylinder, since superconducting currents arise that screen the interior of the cylinder from the field. When the applied field reaches the value of the critical field $B_c(T)$ at the temperature T, the superconducting state ceases to exist because it is energetically more favorable to have the region it occupies filled by the magnetic field. Hence we can calculate the so-called "condensation energy" density of the superconductor as the equivalent magnetic energy density. The energy density associated with a magnetic field (or, which is the same, with the the superconducting current density needed to expel it) is $u_{mag} = \frac{1}{2}\vec{B} \cdot \vec{H} = \frac{B^2}{2\mu_0}$

Here we considered that the normal and superconducting states are basically non-magnetic, so that $\vec{B} = \mu_0 \vec{H}$ (the normal metal is in fact a Pauli paramagnet, but also in this case $\mu_r \cong 1$)

The condensation energy density u_{SC} of the superconductor corresponds to the value of the magnetic energy density when the B field is equal to its critical value:

$$u_{SC} = u_{B_c} = \frac{B_c^2(T)}{2\mu_0}$$

The simple formula for the superconducting condensate energy density is actually only valid for a cylinder. For other geometries the amount of energy required to expel the magnetic field turns out to be equal to the magnetic field energy times a geometrical factor.

Mechanical force in an electric circuit

Consider a closed circuit of resistance R and inductance L, connected to a source that provides a constant current I (with I = const, the power transferred to the electrons is dissipated as heat: $\wp_{\text{charges}} = \wp_{\text{Joule}} = R\,I$). Suppose that we move or deform the circuit in such a way as to vary the inductance *L* and thus the flux through the circuit. The voltage drop across the resistance *R* is:

$\Delta V = \mathscr{E} + \mathscr{E}_{\text{induced}} = \mathscr{E} - \dfrac{d\Phi_B}{dt} = R\,I$ The e.m.f supplied by the source that maintains the current constant must overcome the induced e.m.f. due to he change of *L*, as well as the Joule losses Hence:

$$\mathscr{E} = \frac{d\Phi_B}{dt} + R\,I = \frac{d(L\,I)}{dt} + R\,I = \frac{dL}{dt}I + R\,I$$

On the other hand, by energy conservation, the energy provided by the source must be equal to the change in energy stored in the magnetic field, plus the heat lost by Joule heating, plus the mechanical work *W* to move the coil:

$$\wp_{source} = \mathscr{E}\,I = \frac{dU_{mag}}{dt} + \wp_{\text{Joule}} + \frac{dW_{mec}}{dt}$$

Assuming that the work is done by a force *F* across a distance *dx*, we find:

$$\mathscr{E}\,Idt = d\left(\frac{1}{2}L\,I^2\right) + \delta Q_{Joule} + dW_{mec} = \frac{1}{2}I^2dL + \delta Q_{Joule} + Fdx$$

(note: δQ_{Joule} is heat, <u>not</u> charge)

Plugging the electromotive force found above in the last equation we finally get:

$\mathscr{E}\,Idt = dL\,I^2 + R\,I^2dt = \dfrac{1}{2}I^2dL + \delta Q_{Joule} + Fdx$. Since Joule's heat is $\delta Q_{Joule} = R\,I^2dt$, this gives:

$Fdx = I^2dL - \dfrac{1}{2}I^2dL = +\dfrac{1}{2}I^2dL$. Hence: $\boxed{F = \dfrac{1}{2}I^2\dfrac{dL}{dx}}$

This last expression for the force can also be written, using $U_{mag} = \dfrac{\Phi_B{}^2}{2L}$, as: $\boxed{F = -\left.\dfrac{dU_{mag}}{dx}\right|_{\Phi_B = const}}$

Inductance, energy & force in a magnetic circuit

We saw in topic one that in a magnetic circuit Hopkinson's law applies: $\mathscr{M} = N\,I = \Re\Phi_{\text{core}}$
Here Φ_{core} is the flux through a cross section of the circuit; for a coil of N turns, the total flux crossing it is $\Phi_B = N\Phi_{core}$. Neglecting the magnetic field outside the circuit, we have:

$NI = \Re\Phi_{core} = \Re\dfrac{\Phi_B}{N}$, or: $\Phi_B = \dfrac{N^2}{\Re}I = L\,I$. The <u>self-inductance</u> of a coil of *N* turns wrapped around a magnetic circuit of reluctance $\Re$ is thus: $L = \dfrac{N^2}{\Re}$

The magnetic energy of a magnetic circuit with only one coil around it carrying a current I is :

$$U_{mag} = \frac{1}{2}LI^2 = \frac{1}{2}\frac{N^2}{\Re}I^2 = \frac{1}{2}\frac{\mathscr{M}^2}{\Re} = \frac{1}{2}\Re\Phi_{core}^2 = \frac{1}{2}\mathscr{M}\,\Phi_{core}$$

If there is a gap in the magnetic circuit, there is a force between the opposite poles of the electromagnet across the gap, given by:

$F = \dfrac{1}{2}I^2\dfrac{dL}{dx} = \dfrac{1}{2}I^2N^2\dfrac{d}{dx}\left(\dfrac{1}{\Re}\right) = -\dfrac{1}{2}\dfrac{\mathscr{M}^2}{\Re^2}\dfrac{d\Re}{dx}$, that is: $\boxed{F = -\dfrac{1}{2}\Phi_{core}^2\dfrac{d\Re}{dx}}$

Although we calculated this force in the previous slide assuming a constant current, the force between the poles cannot depend on how the power supply works, but it is always the same for a given pole density, that is, for a given magnetic flux through the core. This is true even if the magnetomotive force is provided by a permanent magnet instead of a coil. Notice that such force can be written, using the expression for the energy given above, as: $F = -\left.\dfrac{dU_{mag}}{dx}\right|_{\Phi_{core} = const}$

Electric machines & the electric grid

Classification of electric machines:

1) Machines that convert mechanical energy/work into electric energy: **GENERATORS**
2) Machines that convert electric energy into mechanical energy/work: **MOTORS**
3) Machines that inter-convert electric energy: **TRANSFORMERS**

In cases 1) and 2), the machine contains at least **a coil moving in an external magnetic field**. Whenever this happens, there is:

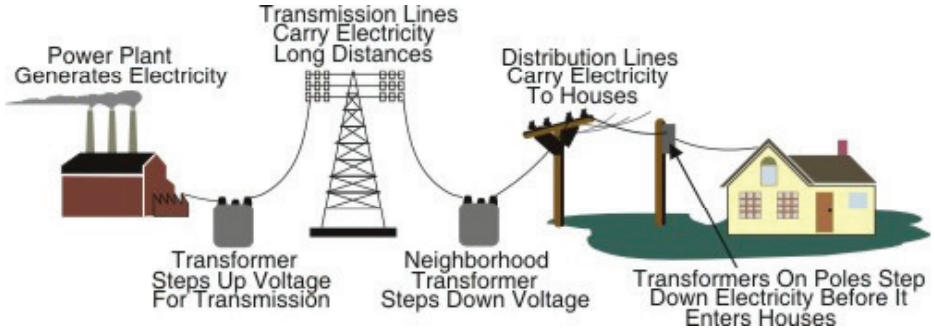
a) an induced e.m.f./back e.m.f. in the coil, as a result of the time-varying flux;
b) a mechanical boost/friction force due to the Lorentz force

It is impossible to have one without the other; however:

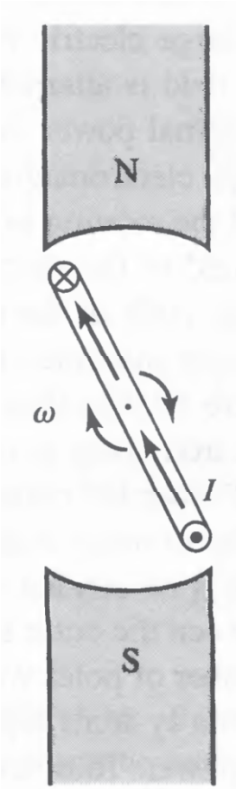
- generators are designed in such a way as (also) minimizing the back e.m.f. that would reduce the output voltage;
- motors are designed in such a way as (also) maximizing the current through the loop.

In both cases part of the task is achieved by ensuring that the magnetic field generated by the current in the loop is negligible with respect to the external field

OVERVIEW of an ELECTRIC GRID (electric power delivery system)

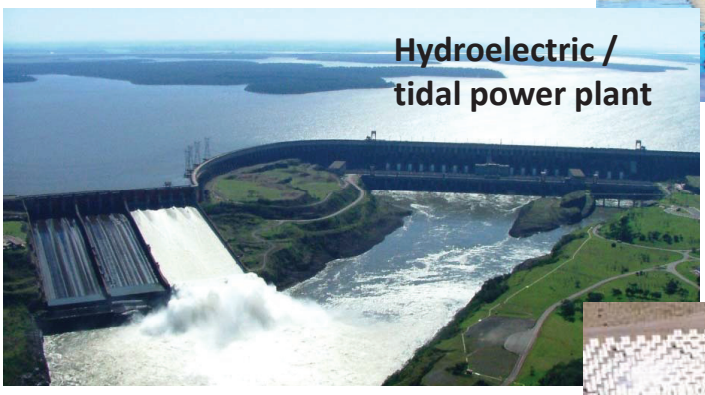


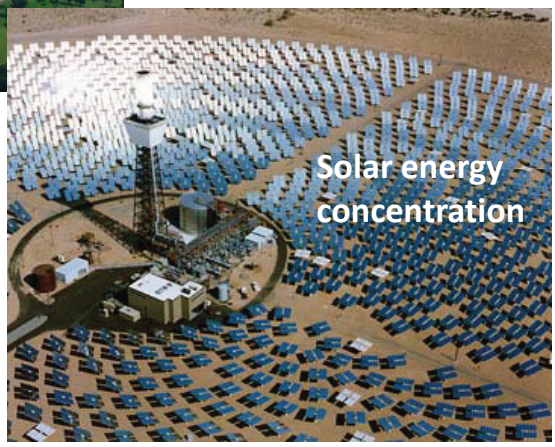
AC generators (power plants)



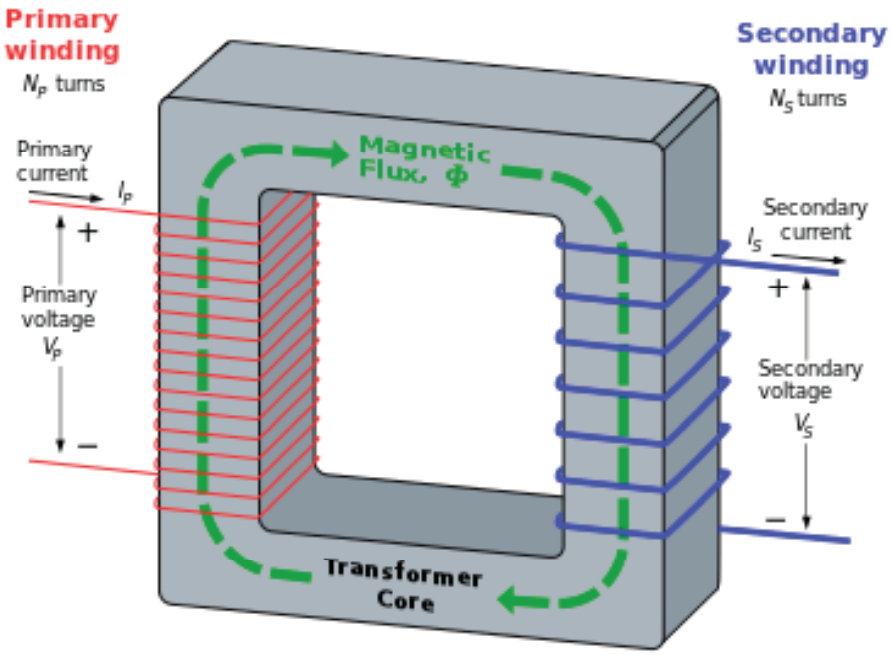
$$\Phi = \int B \cdot dA = BA\cos\omega t$$

$$\Rightarrow \mathscr{E} = -\frac{d\Phi}{dt} = BA\omega \sin \omega t$$





(Ideal) transformer



The primary coil consists of N_P turns, the secondary one of N_S turns. Suppose an AC voltage is applied to the primary coil. By Faraday's law, the induced e.m.f. in the secondary coil is:

$$V_S = N_S \frac{d\Phi}{dt}.$$

Since the same magnetic flux goes through the primary coil, one also must have:

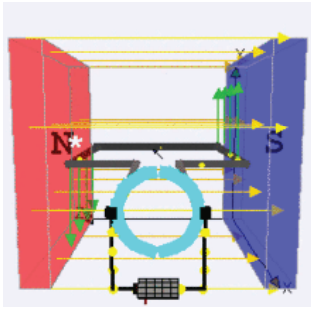
$$V_P = N_P \frac{d\Phi}{dt}.$$

Hence for an ideal transformer $\frac{V_P}{V_S} = \frac{N_P}{N_S} = a$, where a is the winding turns ratio

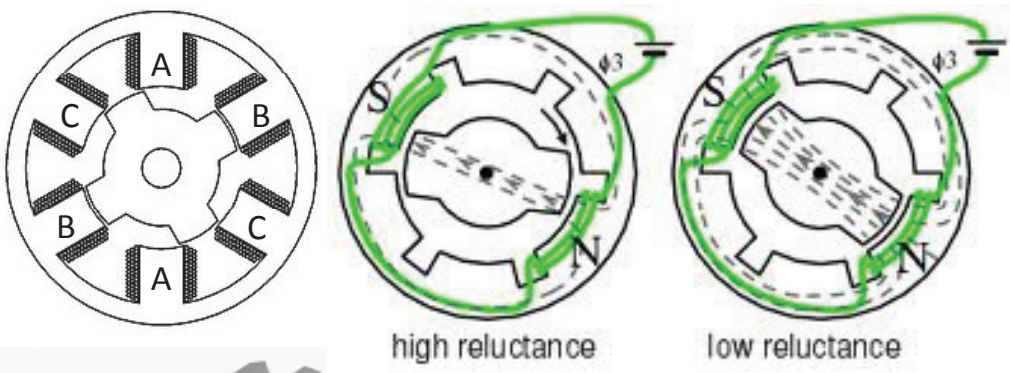
Motors

A motor has a moving part (**rotor**) and a fixed part (**stator**). There are many types of motors:

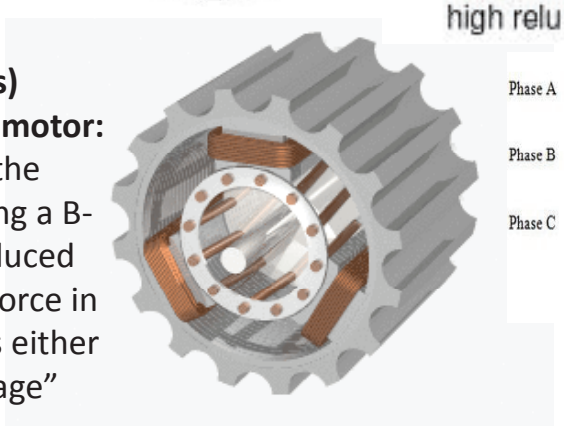
(brushed or brushless)
DC motor: a DC current, whose direction reverses every half-turn, flows through the rotor

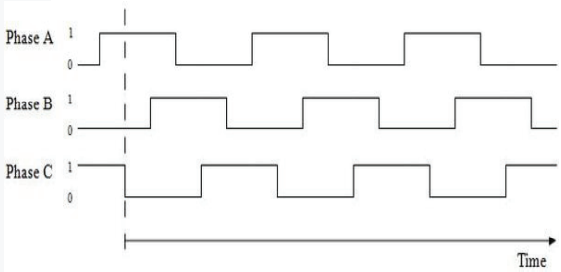


Reluctance motor : AC power (usually with different phases) is supplied to the windings (A, B, C) of the stator. The total reluctance of the magnetic circuit made of stator and rotor depends on their relative orientation



Induction (asynchronous) motor and synchronous motor: AC power is supplied to the stator windings, producing a B-field which causes an induced current and a magnetic force in the rotor, which contains either windings or a "squirrel cage"



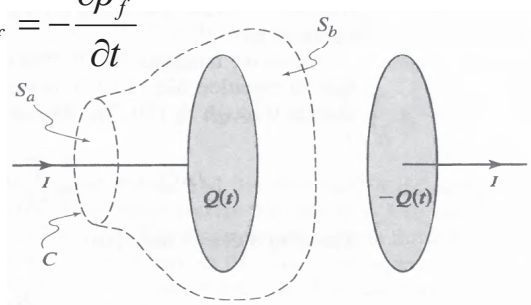


Ampère-Maxwell law & displacement current

Ampère's law $\vec{\nabla}\times\vec{H}=\vec{J}_f$ can't hold in general. In fact :

a) Taking the divergence of Ampère's law we get: $\vec{\nabla}\cdot\vec{J}_f=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=0$. This can't be true always, since it violates the charge conservation law: $\vec{\nabla}\cdot\vec{J}_f=-\dfrac{\partial\rho_f}{\partial t}$
(we see that Ampère's law only holds for *magnetostatics*)

b) Ampère's law in integral form, applied to the charging of a capacitor, gives conflicting results on different surfaces enclosed by the same loop: →



$$\oint\vec{H}\cdot d\vec{\ell}=\int_{S_a}\vec{J}_f\cdot d\vec{a}=I\neq\int_{S_b}\vec{J}_f\cdot d\vec{a}=0 \;\;!!!$$

Maxwell "fixed" Ampère's law by adding an extra term, obtaining: $\boxed{\vec{\nabla}\times\vec{H}=\vec{J}_f+\underbrace{\dfrac{\partial\vec{D}}{\partial t}}_{\text{"Displacement current density"}}}$

a) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{H})=\vec{\nabla}\cdot\vec{J}_f+\dfrac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})=\vec{\nabla}\cdot\vec{J}_f+\dfrac{\partial}{\partial t}\rho_f$ ✓

b) Charging of a capacitor at a constant rate (current): $Q_f(t) = I_f\, t$

$$\oint\vec{H}_1\cdot d\vec{\ell}=\int_{S_1}\vec{J}_f\cdot d\vec{a}+\int_{S_1}\frac{\partial\vec{D}_1}{\partial t}\cdot d\vec{a}=I_f+\varepsilon_0\int_{S_1}\underbrace{\frac{\partial\vec{E}_1}{\partial t}}_{=g\frac{\partial\vec{J}_1}{\partial t}=0}\cdot d\vec{a}=I_f$$

$$D_2=\sigma_f=\frac{Q_f}{S_2}=\frac{I_f\,t}{S_2}\;\Rightarrow\oint\vec{H}_2\cdot d\vec{\ell}=\int_{S_2}\vec{J}_f\cdot d\vec{a}+\int_{S_2}\frac{\partial\vec{D}_2}{\partial t}\cdot d\vec{a}=\frac{d}{dt}(D_2S_2)=I_f\;\Rightarrow H_1=H_2$$ ✓

Displacement current density

displacement current density $\dfrac{\partial\vec{D}}{\partial t}=\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}+\dfrac{\partial\vec{P}}{\partial t}$

You saw the 1st term in Fisica 2, it is called "vacuum" displacement current density

We encountered the 2nd term previously, it is the bound charge current density $\mathbf{J}_b$

When can the displacement current be neglected?

1) In magnetostatics, always (no time-variation of the fields)

2) In general for slow variations (quasi-static approximation):

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}\rightarrow\text{"slow-varying field" approximation}\rightarrow\vec{\nabla}\times\vec{H}(t)=\vec{J}_f(t)$$

3) Especially inside metals, at all frequency attainable in electronics (below infrared/visible):

Example: time-varying current in a metal. If $E=E_0\sin\omega t$, then:

$\vec{J}_f=g\vec{E}$

$$\left.\begin{aligned}J_f&=gE_0\sin\omega t\\ \frac{\partial D}{\partial t}=\varepsilon_0\frac{\partial E}{\partial t}+\underbrace{\frac{\partial P}{\partial t}}_{=0\text{ in a metal}}&=\varepsilon_0\frac{\partial E}{\partial t}=\varepsilon_0E_0\omega\cos\omega t\end{aligned}\right\}\Rightarrow\frac{|\vec{J}_f|_{\max}}{\left|\dfrac{\partial\vec{D}}{\partial t}\right|_{\max}}=\frac{gE_0}{E_0\varepsilon_0\omega}=\frac{g}{\varepsilon_0\omega}\approx\frac{10^{18}}{\omega}$$

$$\Rightarrow\frac{I_{displacement}}{I_{free}}=\frac{\varepsilon_0}{g}\omega\approx10^{-18}\omega$$ → In a metal, normally: $I_{displacement}<<I_{free}$

$\Rightarrow$ the slow varying approximation holds in metals at electronic frequencies

Sources of B and H

The Ampère-Maxwell's equation $\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}$, besides implying the local form of the charge conservation law, together with Gauss's law for H (or B) it entails that there are **four** types of sources of H and B:

$$\begin{cases}\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}=\vec{J}_f+\varepsilon_0\frac{\partial\vec{E}}{\partial t}+\frac{\partial\vec{P}}{\partial t}\\ \vec{\nabla}\cdot\vec{H}=-\vec{\nabla}\cdot\vec{M}\left(=\rho_{mag}\right)\end{cases}\Longleftrightarrow\begin{cases}\vec{\nabla}\times\frac{\vec{B}}{\mu_0}=\vec{J}_f+\underbrace{\vec{\nabla}\times\vec{M}}_{\vec{J}_e}+\underbrace{\frac{\partial\vec{P}}{\partial t}}_{\vec{J}_b}+\underbrace{\varepsilon_0\frac{\partial\vec{E}}{\partial t}}_{\text{vacuum displacement current density}}\\ \vec{\nabla}\cdot\vec{B}=0\end{cases}$$

We see that the sources of B are indeed 4 :

- Free currents, $\vec{J}_f$ (1)
- the curl of M, which we know from Ampère's equivalence theorem corresponds to an equivalent current density $\vec{J}_e=\vec{\nabla}\times\vec{M}$ (2)
- the displacement current density $\partial\vec{D}/\partial t$, which is the sum of :
  - the "proper" (vacuum) displacement current density, $\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (3)
  - the bound charge current density, $\frac{\partial\vec{P}}{\partial t}$ (4)

The last term is not surprising since we already know that $\frac{\partial\vec{P}}{\partial t}=\vec{J}_b$ is the current of bound charges, which is a current of real charges and as such a source of B. Some books define a "total" current density as $\vec{J}_{tot}=\vec{J}_f+\vec{J}_f+\vec{J}_b$, so that one can write an equation similar to the Ampère-Maxwell equation in vacuum you saw in Física 2: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}_{tot}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

Summary: axioms of electromagnetism

in the absence of material media, we get back the equations of Física 2:

$$\begin{cases}\text{I)}\ \vec{\nabla}\cdot\vec{D}=\rho_f\\ \text{II)}\ \vec{\nabla}\cdot\vec{B}=0\\ \text{III)}\ \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}\\ \text{IV)}\ \vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}\\ \text{V)}\ \vec{f}_{Lorentz}=\left(\rho_f+\rho_b\right)\vec{E}+\left(\vec{J}_f+\vec{J}_b\right)\times\vec{B}\end{cases}$$

$$\rho_{tot}=\rho_f,\vec{D}=\varepsilon_0\vec{E}\ \Rightarrow\ \vec{\nabla}\cdot\vec{E}=\frac{\rho_{tot}}{\varepsilon_0}$$

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\begin{cases}\vec{D}=\varepsilon_0\vec{E}\\ \vec{B}=\mu_0\vec{H}\end{cases}\Rightarrow\ \vec{\nabla}\times\vec{B}=\mu_0\left(\vec{J}_f+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

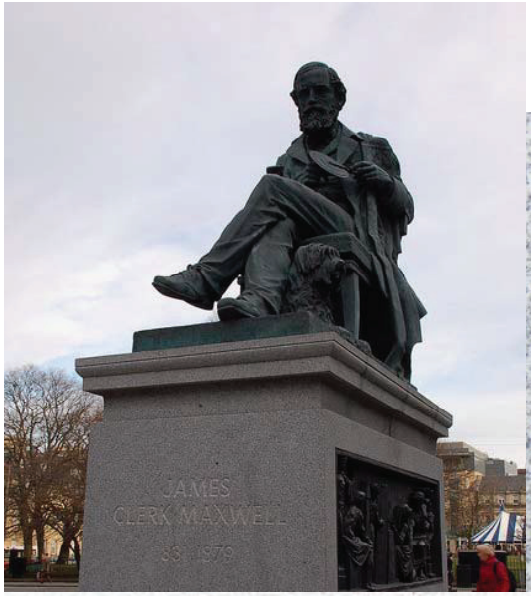
$$\vec{J}_b=0,\rho_b\Rightarrow\ \vec{f}_{Lorentz}=\rho_f\vec{E}+\vec{J}_f\times\vec{B}$$

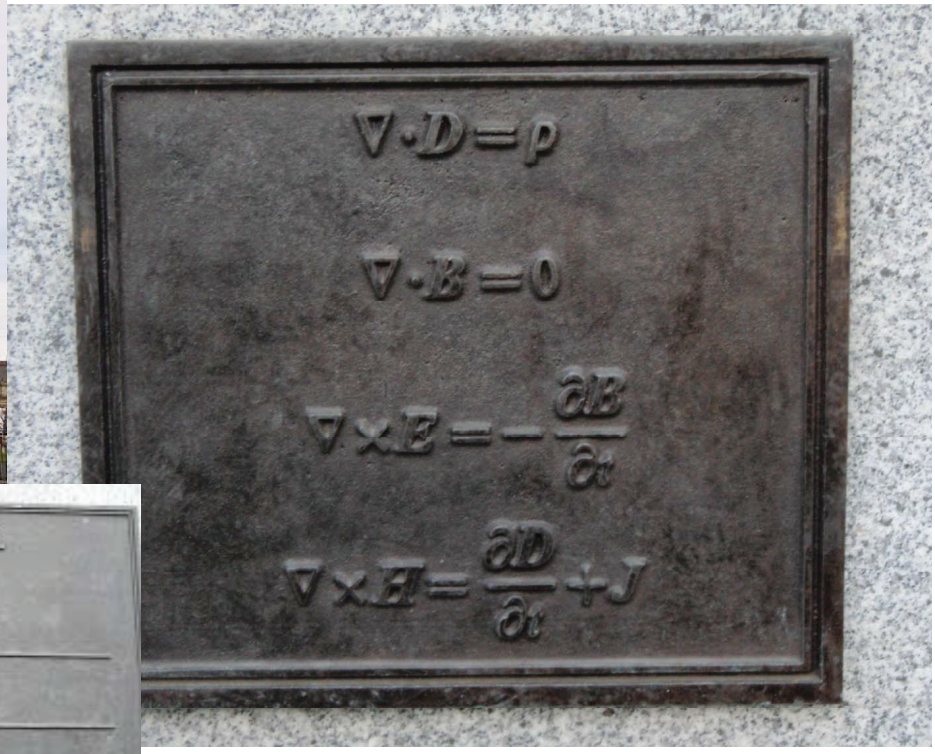
EMAG =
Maxwell's equations
+
Lorentz force
+
Constitutive relations

Some consequences:

→ **B** and **E** fields are interrelated !!!
→ Poynting's theorem (e.m. energy theorem)
→ Existence of e.m. waves !!!
→ Uniqueness theorem for EMAG

MAXWELL's EQUATIONS





George Street, Edinburgh

e.m. energy (Poynting's) theorem in vacuum

If we take the dot product of **E** with Ampère-Maxwell's equation $\frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ we get:

$\frac{1}{\mu_0}\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \varepsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t}$. Using the general vector identity $\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)$, it is :

$$\varepsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{\mu_0}\vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \frac{1}{\mu_0}\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = -\frac{1}{\mu_0}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}\cdot\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)$$

(we also used $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$)

With the definition of Poynting's vector $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$, the last equation can be written as:

$$\varepsilon_0 \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0}\vec{B}\cdot\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\frac{1}{2}\varepsilon_0 \vec{E}^2 + \frac{\partial}{\partial t}\frac{1}{2\mu_0}\vec{B}^2 = -\vec{\nabla}\cdot\vec{S}$$

. Since: $\begin{cases} \frac{1}{2}\varepsilon_0 E^2 \rightarrow \text{vacuum electric energy density} \\ \frac{1}{2\mu_0}B^2 \rightarrow \text{vacuum magnetic energy density} \end{cases}$

$$\Rightarrow \frac{\partial}{\partial t}u_{em} + \vec{\nabla}\cdot\vec{S} = 0$$

→ <u>Poynting's theorem</u> (in vacuum) conservation of electromagnetic energy

In integral form (integrating over the volume) : $\boxed{-\frac{\partial}{\partial t}U_{em} = \int dV\, \vec{\nabla}\cdot\vec{S} = \Phi_{\vec{S}}}$

→ the variation of electromagnetic energy (in vacuum) is equal to the flux of Poynting's vector; Poynting's vector takes e.m. energy across the boundary of a volume

Def: time-average of the modulus of Poynting vector = **irradiance ($\mathfrak{I}$) =**
= (electromagnetic) energy flux per unit area and unit time

Poynting's theorem in linear media

We define the **Poynting's vector** as $\vec{S} = \vec{E} \times \vec{H}$. Taking the divergence of the Poynting vector, we find (using the product rule for derivatives): $\vec{\nabla} \cdot \vec{S} = \vec{\nabla} \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \left(\vec{\nabla} \times \vec{E}\right) - \vec{E} \cdot \left(\vec{\nabla} \times \vec{H}\right)$

Using Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and Ampère-Maxwell's law $\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$ this entails:

$$\vec{\nabla} \cdot \vec{S} = \vec{H} \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right) - \left(\vec{E} \cdot \vec{J}_f + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right) = -\vec{E} \cdot \vec{J}_f - \left(\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right)$$

Since for a linear medium $\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2}\frac{\partial\left(\vec{E} \cdot \vec{D}\right)}{\partial t} + \frac{1}{2}\frac{\partial\left(\vec{H} \cdot \vec{B}\right)}{\partial t} = \frac{\partial\left(u_{el} + u_{mag}\right)}{\partial t}$ where:

$$\begin{cases} u_{el} = \frac{1}{2}\vec{E} \cdot \vec{D} \rightarrow \text{electric energy density} \\ u_{mag} = \frac{1}{2}\vec{H} \cdot \vec{B} \rightarrow \text{magnetic energy density} \end{cases}$$

We then get : $\vec{\nabla} \cdot \vec{S} = -\vec{E} \cdot \vec{J}_f - \frac{\partial}{\partial t}\left(u_{el} + u_{mag}\right)$, or:

$$\boxed{-\frac{\partial u_{e.m.}}{\partial t} = \vec{E} \cdot \vec{J}_f + \vec{\nabla} \cdot \vec{S}}$$

In integral form (integrating over a finite volume) :

→ Poynting's theorem (e.m. energy theorem):

$$\boxed{\frac{\partial U_{em}}{\partial t} = -\wp_{mechanical} - \int d\tau\, \vec{\nabla} \cdot \vec{S}}$$

The loss of electromagnetic energy in a given volume of space is equal to the flux of Poynting's vector (which carries energy out of the volume) plus the power used to accelerate charges. (the latter is equal to Joule's heat for steady currents inside a conductor)

Electromagnetic waves (light) in vacuum

Maxwell's eq. (**in vacuum**)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\Rightarrow \quad \nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\nabla \times \mathbf{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Using the vector identity: $\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$ we get the:

e.m. wave equations

$$\boxed{\begin{cases} \nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \nabla^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2} = 0 \end{cases}}$$

Speed of propagation: $\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \cdot 10^8\, m/s}$

$$c = 1/\sqrt{\left(8.85 \cdot 10^{-12} s^2 C^2 m^{-3} kg^{-1}\right)\left(4\pi \cdot 10^{-7} mkgC^{-2}\right)}$$

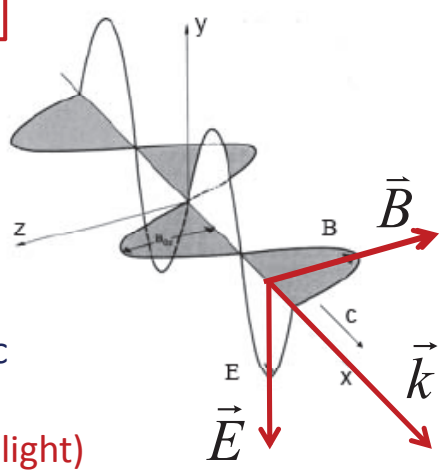
In 1D: $\frac{d^2 y}{dx^2} - \frac{1}{c^2}\frac{d^2 y}{dt^2} = 0$ ➡ harmonic solution: $\boxed{y(x,t) = A\sin(\omega t - kx + \varphi_0) \quad \text{con } \omega/k = \lambda f = c}$

In 3D, ***k*** and ***y*** are vectors: $\vec{Y}(\vec{r},t) = \vec{A}\sin(\omega t - \vec{k} \cdot \vec{r} + \varphi_0)$

$$\left.\begin{array}{l} \vec{\nabla} \cdot \vec{E} = 0 \\ \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{array}\right\} \Rightarrow \vec{k} \perp \vec{E}\ ,\ \vec{k} \perp \vec{B}\ ,\ \vec{B} \perp \vec{E}$$

→ e.m. waves are transverse waves

→ in vacuum, Faraday's law implies B = E/c



the direction of **E** is called polarization of the e.m. wave (light)

E.m. waves in nonmagnetic linear dielectrics with no free charges nor currents (ideal dielectrics)

Maxwell equations for a nonmagnetic, nonconducting medium ($\mu_r = 1$):

$$\begin{cases} \vec{\nabla}\times\vec{H} = \vec{J}_f + \dfrac{\partial\vec{D}}{\partial t} \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\cdot\vec{D} = \rho_f \end{cases} \xRightarrow[\text{(with } \mathbf{J}_f = 0)]{} \vec{\nabla}\times\vec{B} = \mu_0\frac{\partial\vec{D}}{\partial t} \;,\; \vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\mu_0\frac{\partial^2\vec{D}}{\partial t^2}$$

Using the linear constitutive relation for D, we get: $\boxed{\nabla^2\mathbf{E} = \mu_0\varepsilon_r\varepsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}}$

This is the wave equation of waves propagating with speed $v_f = \dfrac{1}{\sqrt{\varepsilon_r\varepsilon_0\mu_0}} = \dfrac{1}{\sqrt{\varepsilon_r}}c = \dfrac{c}{n}$

where $n = \sqrt{\varepsilon_r}$ = **refraction index** and $v_f = \dfrac{c}{n}$ = **phase velocity**

From Faraday's law we get $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \Rightarrow \vec{k}\times\vec{E}_0 = -\omega\vec{B}_0 \quad \Rightarrow \left|\vec{B}_0\right| = \dfrac{k}{\omega}\left|\vec{E}_0\right| = \dfrac{\left|\vec{E}_0\right|}{v_f} = n\dfrac{\left|\vec{E}_0\right|}{c}$

ε_r is the dielectric constant of electrostatics. If ε_r did not depend on frequency, n would be constant. This is only true in the low-frequency limit, while in general <u>***ε* and *n* depend on ω**</u>.

Light as electromagnetic energy: irradiance

Energy density (u) associated with **E** and **B** fields : $u_e = \dfrac{1}{2}\varepsilon_0\varepsilon_r E^2$; $u_m = \dfrac{1}{2\mu_0\mu_r}B^2$ (with $\mu_r \cong 1$)

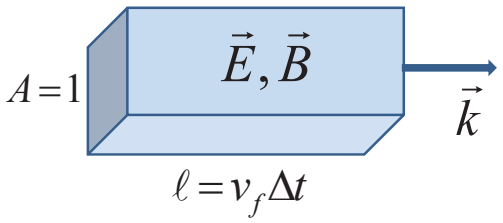
Since $B = \dfrac{E}{v_f} = \sqrt{\varepsilon_r}\dfrac{E}{c}$, it is $u_m = \dfrac{1}{2\mu_0}B^2 = \dfrac{n^2}{2\mu_0}\dfrac{E^2}{c^2} = \dfrac{1}{2}\varepsilon_r\varepsilon_0 E^2 = u_e \Rightarrow \quad u_{e.m.} = u_e + u_m = \varepsilon_r\varepsilon_0 E(t)^2$

Visible light: ω, $\nu \sim 10^{15}$ Hz , the time variation of **E** can't be measured, nor that of u !!! We can only measure time averages such as that of the energy density $\langle u \rangle$. For a harmonic wave:

$$u_{e.m.} = \varepsilon_r\varepsilon_0 E(t)^2 = \varepsilon_r\varepsilon_0 E_0^2\sin^2(\omega t) \Rightarrow \langle u_{e.m.}\rangle = \varepsilon_r\varepsilon_0 E_0^2\langle\sin^2(\omega t)\rangle = \frac{1}{2}\varepsilon_r\varepsilon_0 E_0^2$$

< > = time average

Energy flux divided by area = energy transported by the wave per unit time and area.



For a plane wave in vacuum:

$$\Delta Vol = \ell A = v_f A\Delta t$$

$$\Delta U_{A=1,\Delta t=1} = \langle u_{em}\rangle\Delta Vol_{A=1,\Delta t=1} = v_f\langle u_{em}\rangle = \Phi_u$$

The flux is in the direction of **k**. In vector form : $\boxed{\vec{S} = \vec{E}\times\vec{H} = \Phi_u\vec{k}}$ $\vec{S}$ = **Poynting's vector**

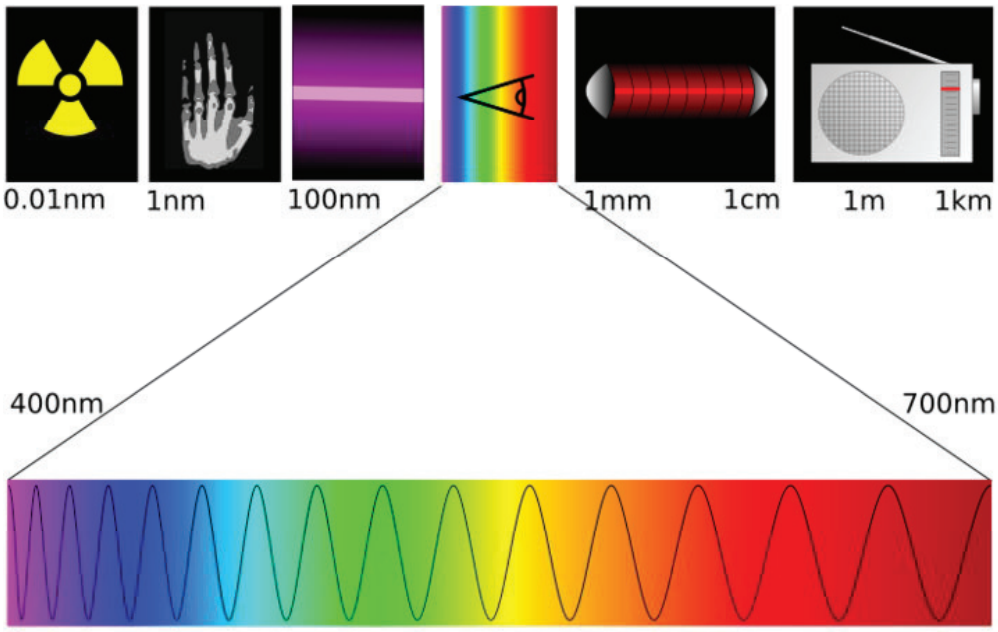
In fact, $\vec{E}\times\vec{H}$ is parallel to $\vec{k}$, and $\left|\vec{S}(t)\right| = EH = EB/\mu_0 = E(t)^2/(\mu_0 v_f) = v_f\cdot\varepsilon_r\varepsilon_0 E(t)^2$

<u>The time-averaged energy crossing a surface per unit time and unit surface</u> is called **irradiance** , and it equals the time average value of the modulus of $\vec{S}$. For a monochromatic e.m. wave:

$$\boxed{\Im = \langle S\rangle = v_f\langle u_{e.m.}\rangle = \frac{1}{2\mu_0 v_f}E_0^2}$$

NOTE: light (e.m. wave) carries not only energy, but also linear and angular momentum

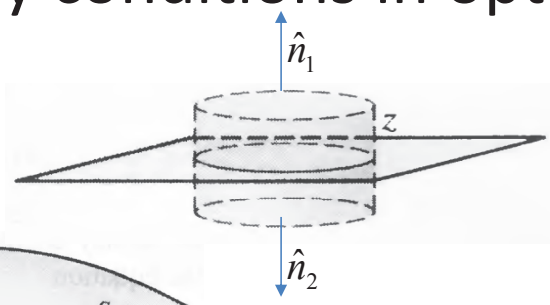
Types of light: spectral ranges



harmonic waves travelling at speed c (or c/n): $\lambda f = c$ (or c/n)
f is a constant $\Rightarrow \lambda$ (inside dielectric) = λ(vacuum)/n = λ_0/n
Wave frequency (and vacuum wavelength) is associated with **COLOR**

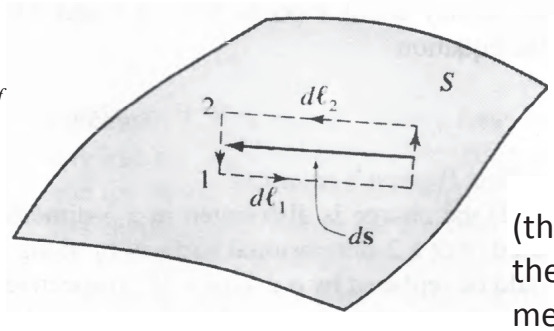
Boundary conditions in optics

$$\begin{cases} \vec{\nabla}\times\vec{H} = \vec{J}_f + \dfrac{\partial\vec{D}}{\partial t} \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\cdot\vec{D} = \rho_f \end{cases}$$



$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\Rightarrow \boxed{B_{n1} = B_{n2}}$$



$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \Rightarrow \quad \boxed{E_{t1} = E_{t2}}$$

(the flux of B through the area da goes to zero in the limit that the segments dℓ_1 and dℓ_2 tend to merge – they are taken just across the boundary)

Normally there is neither free surface charge nor free surface current on the interface. In such case the four boundary conditions that must be satisfied (for linear media) are:

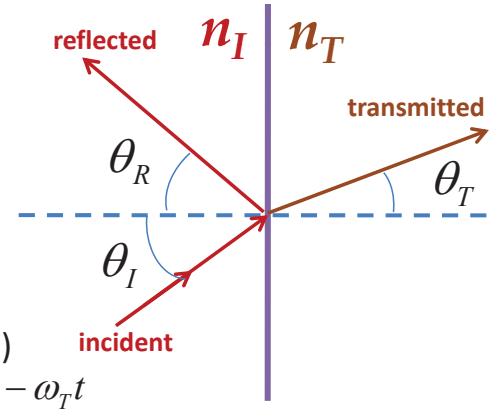
$$\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp} \quad \text{and} \quad B_{1\perp} = B_{2\perp},$$

$$\mathbf{E}_{1\parallel} = \mathbf{E}_{2\parallel} \quad \text{and} \quad \mathbf{B}_{1\parallel}/\mu_1 = \mathbf{B}_{2\parallel}/\mu_2$$

Usually in optics one deals with nonmagnetic media. Even in the case of soft ferromagnets, the magnetization cannot follow applied fields at frequencies higher than few MHz, so that the contribution of M to B at IR or optical frequencies is negligible. Hence the boundary conditions for optical fields become simply:

$\boxed{\mathbf{B} = \text{const}\ ,\ E_{//} = \text{const}\ ,\ \epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}}$

Reflection & refraction at a planar interface



Boundary between media: light is partially reflected (R) and partially transmitted (T) (that is, refracted). The **E** (or **B**) field on the left-hand side of the boundary is the sum of $\mathbf{E}_I$ and $\mathbf{E}_R$, on the other side it is equal to $\mathbf{E}_T$

Boundary conditions in optics: **B** = const, $E_{//}$ = const

→ There exists a fixed relation between the fields at all points $\vec{r}$ of the boundary. For a monochromatic incident wave, this implies that the three waves (incident, reflected & transmitted) must have equal total phase : $\vec{k}_I \cdot \vec{r} - \omega_I t = \vec{k}_R \cdot \vec{r} - \omega_R t = \vec{k}_T \cdot \vec{r} - \omega_T t$

Setting the origin $\vec{r} = 0$ at a point on the boundary, at the origin $-\omega_I t = -\omega_R t = -\omega_T t$, hence

$$\boxed{\omega_I = \omega_R = \omega_T}$$

(consistent with the definition of photon and with energy conservation)

For $t = 0$: $\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r} \Rightarrow \vec{k}_I$, $\vec{k}_R$ and $\vec{k}_T$ are coplanar

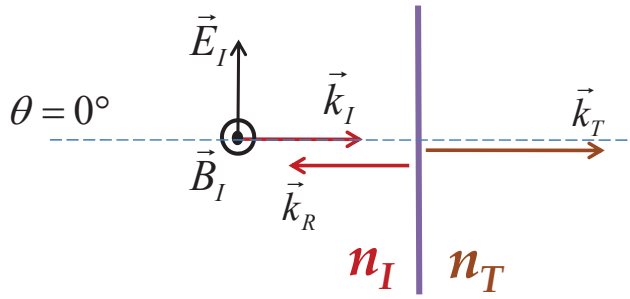
Taking $\vec{r}$ to be coplanar with the wave vectors we get: $\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} \Rightarrow k_I r \sin\theta_I = k_R r \sin\theta_R$

This implies: $k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$

⇒ **law of specular reflection** $\boxed{\theta_R = \theta_I}$ (incident and reflected waves propagate in the same medium of refractive index n_I → they have the same value of $|\boldsymbol{k}|$)

⇒ **Snell's law** $\boxed{n_T \sin\theta_T = n_I \sin\theta_I}$ (the transmitted wave propagates in medium with $n_T \neq n_I$ with same frequency as the incident wave → λ changes)

Reflectance for normal incidence



Boundary conditions in optics: **B** = const, $E_{//}$ = const .

For light impinging at normal incidence on a planar boundary this implies:

$$\begin{cases} E_{0I} + E_{0R} = E_{0T} \\ B_{0I} + B_{0R} = B_{0T} \end{cases}$$

From Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow kE_0 = \omega B_0 \Rightarrow B_0 = \frac{E_0}{\omega/k} = n\frac{E_0}{c}$$

$$\Rightarrow \begin{cases} E_{0I} + E_{0R} = E_{0T} \\ n_I E_{0I} - n_I E_{0R} = n_T E_{0T} \end{cases}$$

The minus sign takes into account the change in relative orientation of E & B in the reflected wave (as $\vec{k}_R = -\vec{k}_I$)

Solving for the transmitted and reflected field amplitudes:

$$\begin{cases} E_{0T} = \dfrac{2n_I}{n_I + n_T} E_{0I} \\ E_{0R} = \dfrac{n_I - n_T}{n_I + n_T} E_{0I} \end{cases}$$

We thus get:

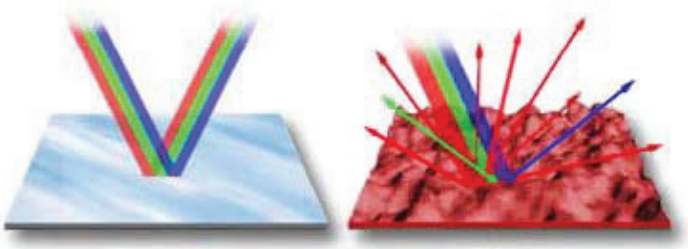
$$\frac{E_{0R}}{E_{0I}} = \frac{n_I - n_T}{n_I + n_T} \Rightarrow \boxed{R = \frac{\Im_R}{\Im_I} = \frac{|E_{0R}|^2}{|E_{0I}|^2} = \left|\frac{n_I - n_T}{n_I + n_T}\right|^2}$$

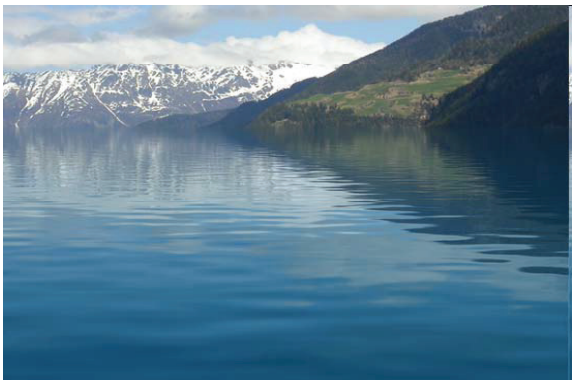
REFLECTANCE

The transmittance T is defined as $T = \dfrac{\Im_T}{\Im_I}$. For energy conservation, it must be: $R + T = 1$

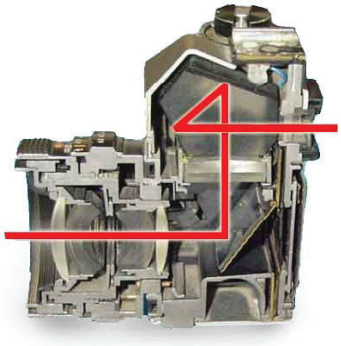
NOTE: reflectance and transmittance are the same if light impinges from one side or the other. For the air-glass interface (e.g. at a clean window), R is about 4% at normal incidence, so a window reflects at least 8% of the incoming light (R increases with increasing incidence angle)

Specular reflection *vs* Diffuse reflection



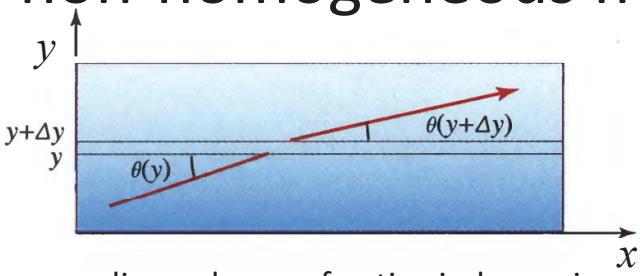


Total internal reflection: optical fibers, pentaprism

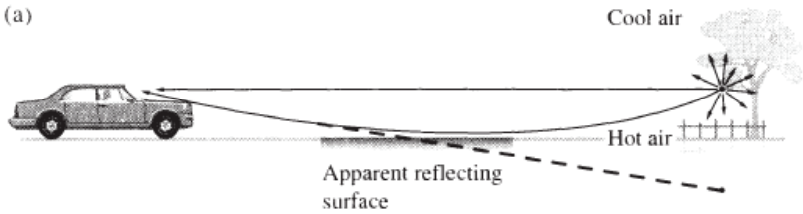


http://www.youtube.com/watch?v=hBQ8fh_Fp04&feature=related

Refraction in non-homogeneous media: mirages



Consider an inhomogeneous medium whose refractive index varies with the vertical coordinate y. A light ray undergoes refraction wherever there is a change of refractive index. Due to Snell's law, since n is varying continuously with y, a light wave travelling through this medium undergoes a continuous refraction → **mirages:**



http://www.youtube.com/watch?v=d8zUHBbflmM

GRIN waveguide

This effect leads to the phenomenon of mirages, and is also exploited commercially to achieve so-called graded-index materials (GRIN), used for lenses, waveguides and fibers.